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Making Mathematics Memorable, Meaningful, and Fun: Activities to Enhance Precalculus

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MAKING MATHEMATICS MEMORABLE, MEANINGFUL, AND FUN:

ACTIVITIES TO ENHANCE PRECALCULUS

by

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ABSTRACT

MAKING MATHEMATICS MEMORABLE, MEANINGFUL, AND FUN:
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Nathaniel B. White

The University of Wisconsin-Milwaukee, 2015
Under the Supervision of Professor Gabriella Pinter

To master material, students need to make it their own. As teachers, we should structure their interactions with mathematics in ways that are memorable, meaningful, and fun. One way to do this is to provide activities that stretch beyond the textbook and lead students to think and talk to one another about mathematics. This thesis contains a set of activities designed to enhance a precalculus course, along with solutions and feedback on each activity.

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ACKNOWLEDGEMENTS

I. Introduction and Context

To fully comprehend material, students need to engage with it. Ideally, students will be able to encounter new content and learn the ideas and mathematical processes while thinking about contextual examples, rather than through rote repetition of exercises. This project includes a set of activities for students at the precalculus level that goes beyond traditional textbook material. These activities are designed to contribute to many levels of student learning. First, by staying with an activity for several days, students will see that many building blocks come together to allow mathematicians to answer questions that are more involved than any single textbook exercise. Second, these activities will allow students to think about how to use the mathematical skills they are developing, not just to memorize formulas and regurgitate them on many questions very similar to the examples. Third, these activities should introduce students to mathematical concepts beyond those driven by calculus and engineering needs. Fourth, some of these problems provide a way to build skills related to the use of technology in solving problems. Underlying all of this work is the need for students to read the handouts to gain an understanding of a problem and the information they may need to solve it. In my own classroom, I have found that students are rarely good technical readers. By moving some of the guidance that I could provide during class into the handouts, my goal is for students to read and to try to make sense of their reading in order to move on to the next piece of the problem. I am far from alone in the push to engage students with these more involved activities in addition to textbook exercises and basic skill building. There is both national and local context for the work presented here, and a quick internet search will turn up many examples.

Regardless of the content covered in a particular mathematics course, there are process goals and outcomes that should drive the learning. Both the National Council of Teachers of Mathematics (NCTM) and the Common Core State Standards (CCSS) specify process goals. For

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the NCTM, there are five strands of process: problem solving; reasoning and proof; communication; connections; and representation (details can be found at http://www.nctm.org/Standards-and-Positions/Principles-and-Standards/Process/). The Common Core State Standards take these strands and build on them to create statements about mathematical practice. We, as math educators, should seek to help our students: make sense of problems and persevere in solving them; reason abstractly and quantitatively; construct viable arguments and critique the reasoning of others; model with mathematics; use appropriate tools strategically; attend to precision; look for and make use of structure; look for and express regularity in repeated reasoning (see http://www.corestandards.org/Math/Practice/).

Within the math department at The Hotchkiss School, where I teach, we have been focused on updating the traditional high school curriculum to meet the needs of the coming century. As a private school, we are not beholden to the CCSS, but many of our ideas are similar to those included in the CCSS. Because we would like students to think about their math experience as a coherent examination of multiple topics rather than a collection of course titles, we have chosen to highlight seven strands that appear throughout the high school math curriculum: functions; iteration and recursion; transformations; proof and reasoning; algebraic properties and notation; statistics; and shape and space (FITPASS). We make a point of talking about each strand in each course and highlighting FITPASS as the structure connecting their work over the span of the high school curriculum. In addition to these seven strands, I have tried to incorporate the use of technology to enhance understanding of concepts in most of the activities. Where possible, this also includes the algorithmic thinking necessary to program a machine. The department here began its curricular overhaul 6 years ago with its Intermediate Algebra course, a hybrid between traditional Algebra I and Algebra II courses, and we have been

working our way up towards Precalculus ever since. 2015-2016 is the first year that almost every unit of study in Precalculus will have an anchor activity. The work done for this thesis includes both new activities and revisions to activities that my colleagues and I have previously created and used in our Precalculus classes.

II. Process

To design a course with activities in mind, I worked first with the idea of backward design espoused in *Understanding by Design* by Grant Wiggins and Jay McTighe. My goal is to balance development of the skills needed for success in calculus with the inclusion of topics that a capable high school student should be exposed to, including dynamical systems and discrete topics such as probability. I also feel there is value in students having a textbook as a resource, so I have designed the course around the 6th edition of the Stewart, Redlin, and Watson textbook *Precalculus, Mathematics for Calculus*. This book is relatively new to our department, though the department did use the 4th edition prior to my arrival. In my 6 years here, we have taught precalculus courses using Dick Brown's *Advanced Mathematics, Precalculus with Discrete Mathematics and Data Analysis*; *Functions Modeling Change: A Preparation for Calculus* by Connally, Hughes-Hallett, Gleason et al; and Foerster's *Precalculus with Trigonometry*. I have arranged the topics in a way that takes into account the order of the current textbook and the timing of our semesters. With the general structure provided by the textbook, I looked for ideas for activities to engage students. This search for ideas included my own past work and studies, ideas from colleagues and conferences, textbook resources, and internet searches.

Once I had topics in mind, I wrote drafts of new activities, edited existing activities, and solicited feedback from colleagues. This often meant a quick trial run during a half-hour course team meeting in which colleagues also teaching Precalculus this year asked questions about an

activity and the ideas covered in it. These questions were very helpful to me in clarifying the goals for each activity and in adjusting the presentation and wording of the activity. For some of the activities, the process was a little different only because I, and sometimes others, had used a different version previously. In the case of the fractal coastline activity, for example, 2 or 3 years ago Marta Eso and I significantly overhauled an activity created by a former colleague, Ron Armontrout. This year, I updated Part C of the activity, which connects most directly to the textbook content.

At the conclusion of each activity, as part of the iterative design process, I asked my students to answer four questions:

- What big ideas did you take away from the lab?
- Did anything surprise you as you went through the lab?
- What ideas are you still confused about?
- What parts of the reading in the handout were unclear?

These questions allow students to reflect on what they've learned and to recognize where they need more help. The answers allow me to gauge student understanding and to refine the activities for next year, a process I will take on in the summer. I have included the responses gathered so far in Appendix J. Because the school year is not yet half over, I have student responses only for the dynamical systems and fractal coastline activities. At the time of this writing, we are finishing the conditional probability unit. The remaining activities have not yet been through this process of soliciting feedback from colleagues, using the activities with my class, and gathering feedback from my students. This work will continue as we progress through the year.

III. Course Outline

This section lays out the course order and the topics covered, including the relevant portion of the textbook. Each of our courses meets for three 45-minute periods and one 85-

minute period per week. The first semester (September through December) is significantly shorter than the second semester (January through early June). Most of the students in this course will take AP Calculus AB the following year, though some will graduate from high school and others will enroll instead in our non-AP calculus course. I have included some work on matrices that would be included in the ideal precalculus course, though our timeline has meant that we have chosen not to cover matrices for the last couple of years. Every precalculus student will take the same semester and final exam, so the teachers must all agree on the content to cover.

Semester 1:

1. <u>Algebra Review</u> (skills check worksheets)

HANDOUTS: MATHEMATICAL THINKING AND HABITS (Appendix A)

- o Factoring
- o Algebraic fractions
- o Solving equations and inequalities, including absolute value equations and inequalities
- o Functions and function notation

2. <u>Functions</u> (Chapter 2)

ACTIVITY: DYNAMICAL SYSTEMS (Appendix B)

- o Definition of a function; graphically: vertical line test
- o Toolkit functions: linear, absolute value, power, root, reciprocal, exponential
- o Piecewise defined functions
- o Properties of functions, finding these graphically
  - o Domain, range, asymptote
  - o Increasing/decreasing
  - o Local maximum/minimum, local maximum/minimum value
- o Average rate of change of a function
  - o Linear functions have constant rate of change
- o Transformations of functions

- o Vertical and horizontal shifting

- o Reflecting graphs in the coordinate axes

- o Vertical and horizontal stretching and shrinking

- o Combinations of the above, order of transformations may matter

- o Odd and even functions

    - o Graphically

    - o Algebraically

- o Algebra of functions: sums, differences, products and quotients of functions

- o Composition of functions

    - o Finding the formula for

    - o Domain of composite functions

    - o Decomposition of functions (breaking apart into the original pieces)

- o One-to-one functions, horizontal line test

- o Inverse functions

    - o Definition, existence

    - o Inverse function property: function composed with its inverse is the identity function

    - o Finding the inverse graphically

    - o Finding the inverse algebraically

- o Modeling with functions (See: Focus on Modeling section after Chapter 2)

3. <u>Polynomial and Rational Functions</u> (Chapter 3)

HANDOUTS: ASYMPTOTES, INEQUALITIES (Appendix C)

- o Quadratic Functions: writing equations from given information; converting between forms

    - o General form: $y = ax^2 + bx + c$

    - o Standard form: $y = a(x-h)^2 + k$

    - o Factored form: $y = a(x-r)(x-s)$

- o Finding the maximum/minimum value of a quadratic function

    - o By completing the square

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- o   Using the line of symmetry
- o   Modeling with quadratic functions
- o   Using the quadratic formula to solve quadratic equations
- o   Polynomial functions: terms and coefficients, degree, leading term
  - o   End behavior of a polynomial function
  - o   Real zeroes of a polynomial function, behavior around zeros (multiplicity)
  - o   Equation to graph and graph to equation
  - o   Local extrema of polynomial functions, number of turning points
  - o   Inflection points
- o   Long division of polynomials, synthetic division
- o   Zeros of polynomials, factoring a polynomial into linear and irreducible quadratic terms
  - o   Remainder and Factor Theorems
  - o   The Rational Zeros Theorem
- o   Solving polynomial equations and inequalities
- o   Complex numbers
  - o   Definition, conjugate
  - o   Operations (add/subtract, multiply, divide) over complex numbers
  - o   Powers of i
  - o   Manipulating expressions with square roots of negative numbers
  - o   Utilizing the complex conjugate to simplify
  - o   Graphing complex numbers on the complex plane
- o   Complex zeros of polynomials
  - o   In the universe of complex numbers, a polynomial can be factored completely into linear factors. A polynomial of degree n has exactly n zeros (counting multiplicities).
  - o   Conjugate Zeros Theorem: complex roots of a polynomial with real coefficients come in conjugate pairs. Thus every polynomial with real coefficients can be factored into linear and irreducible quadratic terms.
  - o   Reconstructing the equation of a polynomial given some of the roots, intercepts, degree

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- o Transformations of the parent reciprocal function, locating asymptotes and intercepts, domain and range
- o Rational functions
  - o Domain question: vertical asymptotes (and their multiplicities) and holes
  - o Zeros and behavior around zeros (multiplicities)
  - o y-intercept
  - o Identify end behavior using polynomial division or by comparing the degrees of the numerator and denominator
  - o Range questions: end behavior, including horizontal and slant asymptotes
- o Sketching the graph of a rational functions
  - o Factor numerator and denominator, identify holes
  - o Identify vertical asymptotes and behavior around these
  - o Identify zeros and behavior around these
  - o Identify y-intercept
  - o Identify end behavior
  - o (If needed, find a test point that is not along the x-axis)
- o Reconstructing a possible equation for the rational functions given its graph
- o Solving rational function inequalities

4. <u>Exponential and Logarithmic Functions</u> (Chapter 4)

HANDOUTS: PROPERTIES, REVIEW

ACTIVITY: FRACTAL COASTLINE (Appendix D)

- o Rules of exponents (review)
- o Exponential functions $y = a \cdot b^x$
  - o Domain, range, asymptote, y-intercept
  - o Increasing/decreasing depending on the value of b
  - o Equation to graph, graph to equation
- o The definition of e, the natural exponential function
- o Definition of the logarithmic functions as the inverse of exponential functions
  - o Converting between exponential and logarithmic form

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- The common and natural logarithm
  - Properties and laws of logarithms
    - Inverse property
    - Product, quotient, power rules
    - Change of base formula
  - Expanding and combining logarithmic expressions
  - Solving exponential equations: isolate the exponential term, take logs, use the power rule to "bring down the exponent," solve for variable
  - Exponential and logarithmic equations of quadratic type
  - Solving logarithmic equations: isolate the logarithmic term (might need to combine logs), write into exponential form, solve for the variable
  - Modeling with exponential and logarithmic functions
    - Compound interest
      - Repeated compounding (principal, annual percentage rate, compounding frequency), formula
      - Annual percentage yield (or effective rate)
      - Continuously compounded interest
    - Exponential growth and decay
      - Equations given doubling time or half-life
      - Equations given relative growth/decay rate (a.k.a. continuous growth/decay rate): "Pert"
    - Newton's Law of Cooling
    - Logarithmic Scales (Richter, pH, decibel)

5. <u>Probability</u> (Chapter 14 and handouts)

HANDOUTS: INTRODUCTION, PROBABILITY WITH AND/OR, WRAP-UP

ACTIVITY: CONDITIONAL PROBABILITY AND MEDICAL TESTING (Appendix E)

- Basic vocabulary: random experiment, trial, outcome, sample space, event, event space, equally likely outcomes

- o Definition of probability as the ratio of the number of favorable outcomes to the total number of outcomes
- o Counting Principles: addition (or), multiplication (and, consecutive choices), complement (at least, at most)
- o Counting permutations: order matters. With and without replacement, with repeated elements, circular permutations
- o Counting combinations: order doesn't matter.
- o Evaluating $_nP_r = P(n,r)$, $_nC_r = C(n,r) = \begin{pmatrix} n \\ r \end{pmatrix}$
- o Mutually exclusive events don't happen at the same time: $P(A \text{ and } B) = 0$. Events that are not mutually exclusive are called overlapping.
- o Conditional probability: $P(A|B)$. The probability of A happening given that B already happened
- o Independent events: the occurrence of one event does not affect the probability of the other event: $P(A) = P(A|B)$ and $P(B) = P(B|A)$. Events that are not independent are called dependent.
- o Conditional probability and Medical Testing: given what percentage of the population carries a genetic marker and given the accuracy of a test to detect the presence of the marker, compute the probability of false negative and false positive answers.
- o Expected value

Semester 2:

6. <u>Probability Continued</u>

ACTIVITY: GEOMETRIC DISTRIBUTION PROJECT (Appendix E)

- o Binomial Probability
- o Binomial and Geometric Distributions

7. <u>Trigonometry</u> (Chapters 5-7)

ACTIVITIES: PREDATOR/PREY and MOVIE SEATING Discovery Projects (Appendix F)

- o Six trig functions:
  - o Degrees vs. radians
  - o Values of all six trig functions: unit circle definition; based on x,y,r; using reference angles, coterminal angles; through a point in the plane;
  - o Using special right triangles, reference angles to generate known values
- o Graphs: tangent, including shifts; full for sine and cosine (shifts and dilations in both horizontal and vertical directions), secant, cosecant, cotangent basics.
  - o Equation to graph and graph to equation
- o Modeling with sinusoids
  - o General setup
  - o Solving graphically with intersection feature, symmetries
  - o solving algebraically with inverse trig
- o Inverse trig functions
  - o domains and ranges
  - o graphically
  - o interpreting the answer given by calculator
- o Solving trig equations
  - o factoring, quadratic forms, applying properties, dealing with multiple angles (e.g. $\sin(3x)=1/2$)
- o Verifying identities: using Pythagorean, reciprocal, quotient properties; addition/subtraction of angles; double-angle formulae
- o Solving any triangle
  - o Law of sines
  - o Law of cosines
- o Areas of triangles and quadrilaterals
- o Navigation problems

8. Polar Coordinates, Complex Numbers, and Parametric Equations (Chapter 8)

ACTIVITY: ROADRUNNER AND COYOTE, possibly LINES 3 WAYS (Appendix G)

- o Polar coordinates

    o   Plotting points and graphs in polar coordinates

    o   Converting points and equations from polar to rectangular and vice versa

o   Complex numbers

    o   Graphing in the complex plane, a+b*i* form.

    o   Representing complex numbers in polar form $rcis(\theta)$

    o   DeMoivre's Theorem and powers and roots of complex numbers

o   Parametric Equations

    o   Eliminating the parameter

    o   Projectile motion

9. <u>Vectors</u> (9.1 and 9.2)

o   Geometric representation of vectors. Adding, subtracting, multiplying by a scalar.

o   Given components of two dimensional vectors find:

    o   Length

    o   A unit vector in its direction

    o   A scalar multiple

    o   Find the sum and difference of two two-dimensional position vectors

o   Find the dot product of two dimensional vectors

    o   Use dot product to find angle between vectors

    o   Parallel, perpendicular vectors

o   Wind, current, navigation problems

o   If time permits, expand to include:

    o   3-D vectors

    o   Cross product

    o   Projection vectors and work

        ▪   Least-squares regression as projection vectors, not a calculus optimization problem

10. <u>Conic Sections</u> (11.1, 11.2, 11.3, 11.4)

ACTIVITY: FERRIS WHEELS (Appendix H)

- o Cartesian equations
    - o Completing the square
- o Geometrically:
    - o Defined by distances (equal from 1 point, equal from point and line, sum of distances from 2 points is constant, difference of distances from 2 points is constant)
    - o Pythagorean property of key distances in ellipse and hyperbola
- o Parametrically?

11. <u>Limits and Continuity</u> (13.1, 13.2, 13.4)

- o Limits
    - o Formal definition and notation
    - o At infinity and at a point (including one-sided)
    - o Evaluating graphically, numerically, algebraically
    - o Return to rational functions
- o Continuity
    - o Definition and choosing values in piecewise functions

12. [if time] <u>Matrices</u> (Chapter 10)

ACTIVITY: WATERSHED (Appendix I)

- o I'm leaving out specific topics, since we have not included matrices at this level during the last 4 years and 2 textbooks.

IV. The Activities

This section includes the goals of the activity for each unit. The activities and the answer key for each are included in the appendices. This section also includes reflection on each activity and possible revisions and extensions.

The activities are written based on the technology we use. All of our students use the TI-Nspire CAS calculators. They also have school-issued Mac laptops on which they can access the

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Nspire Student software when on the school network. Math faculty members have TI-Nspire CX CAS calculators and the Nspire Teacher software installed on their laptops. Every classroom is equipped with a projector, and all school-issued laptops can connect wirelessly to the projector.

1. <u>Algebra Review</u> (skills check worksheets)

HANDOUTS: MATHEMATICAL THINKING AND HABITS (Appendix A)

We opened the year with a focus on mathematical thinking habits in class and algebra review for homework. I created three handouts, each with a different focus, based on the Focus on Problem Solving portion of the stewartmath.com website that accompanies our textbook. Students worked in partners or groups of 3 on the problems in class, and then completed algebra skill review handouts for homework. In my teaching, I have been trying to move rote work to homework and work on the more involved and challenging problems in class. The primary goal of this section of the course is to get students actively thinking and talking to their peers about how to solve a problem that does not look exactly like an example problem. A secondary goal of this set of exercises is to put all students, no matter the differences in background before they started the course, on equal footing with problems that challenge them mentally without exposing differences in technical skills. The third goal, perhaps the most obvious, is to give students a chance to refresh specific algebra concepts that may be rusty or weak. The habits and thinking handouts were well-received by faculty members teaching other sections of the course and the students engaged in solving the problems, but I did not gather specific feedback from the students about the handouts. The skills check worksheets are taken directly from our Advanced Algebra and Trigonometry course, the precursor course to this one.

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2. <u>Functions</u> (Chapter 2)

ACTIVITY: DYNAMICAL SYSTEMS (Appendix B)

We moved from these opening thinking and review problems into the textbook to develop understanding of functions, function notation, transformations and combinations of functions (including composition), and inverse functions. During this work, we also introduced the notion of average rate of change to preview the work of calculus. At the conclusion of the functions unit, we spent 4 class periods working through the Dynamical Systems activity that I wrote. The activity is geared around iterating quadratic functions. Most students in our program had done a similar activity last year in their advanced algebra/trig course on iterating linear functions, so that this was not the first time that they had considered iteration, though their previous work focused on sequence notation.

The Dynamical Systems activity has a mix of content and process goals, more geared toward mathematical thinking than towards calculus. The inspiration for the activity came from work done in MATH 405 Math Models and Applications, my desire for an extension to a lab done in the previous course on iterating linear functions, and the inclusion of this type of equation in Ian Stewart's *In Pursuit of the Unknown: 17 Equations that Changed the World*. This activity exposes students to a process/approach to defining a problem implicitly/recursively, using technology to iterate, creating a time-series plot to see long-term trends, creating a cobweb plot to help explain these long-term trends mathematically, and, ideally, the creation of an explicit/closed-form model. It also illustrates the differences between discrete and continuous systems while introducing the logistic model for restricted growth.

There are several areas in which I would like to improve the activity before using it again next year. The section that talks about the differences between discrete and continuous systems

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does not have much detail; I would like to add more. I would like to flesh out the repeated composition of functions for future iterations of this activity. In particular, I would like to tie it into the work with the cobweb diagram. I would also like to include more background for teachers and develop some extensions that might be used with honors classes, such as working to create the explicit model from the difference equation. I would like to explicitly preview for students the difference equation as a discrete analog to a differential equation. In general, I like to provide my students with a look forward into mathematical tools they will develop in later courses and the types of questions they can answer with those tools. Another possible extension is the use of other non-linear iterated models, such as the Ricker model. I am also considering ways to give students and teachers additional background into the branching points and Feigenbaum's work on chaotic systems.

Because the activity includes content that is not typically taught at the high school level, and because the activity came early in the year, I was able to get only one colleague to use the activity in her class. The combination of new content and different tools is enough to prevent many teachers from trying this and other activities in their classrooms. The hesitation of teachers to utilize these materials is a significant obstacle, and one I intend to commit some of my career to trying to overcome. Because I believe these activities help students meet the learning outcomes described in the opening section, figuring out what support teachers need to use the activities in their own classrooms is critical.

3. <u>Polynomial and Rational Functions</u> (Chapter 3)

HANDOUTS: ASYMPTOTES, INEQUALITIES (Appendix C)

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For this section, I did not create any activities. There are plenty of concrete examples in the textbook, and students need to build basic skills, especially with rational functions, before they are ready to apply what they have learned. Students will see rational inequalities applied in their later work on probability, notably in the medical testing activity. I created the handout on asymptotes to summarize rules for my students. Many precalculus textbooks utilize the multiplicity of zeros when talking about the behavior of polynomials, but they leave out the multiplicity of the zeros of the denominator when talking about behavior around vertical asymptotes. I address this in my class and in the handout.

The Stewart textbook places solving polynomial and rational inequalities in the section on algebra review. Rather than working on these problems early on, we come back to solving them at the conclusion of our section on polynomial and rational functions. The handout summarizes the textbook approach and reminds students that they have developed more tools to deal with these inequalities through their work on polynomial and rational functions.

4. <u>Exponential and Logarithmic Functions</u> (Chapter 4)

HANDOUTS: PROPERTIES, 2 REVIEW HANDOUTS

ACTIVITY: FRACTAL COASTLINE (Appendix D)

The fractal coastline activity was initially created, as noted above, by my former colleague, Ron Armontrout. It was not used for several years, and then Marta Eso and I rewrote the lab to fit with the Foerster textbook in the fall of 2012. While parts A and B of the activity remained similar this year, I rewrote part C to focus more explicitly on content and process goals. The fractal coastline applet shows students another example of how repetition of a simple rule can yield interesting results. Parts A and B ask students to look for patterns in pictures, data, and

graphs; to think about what the extreme cases would look like (roughness of 0, roughness of 1); and to create spreadsheets and scatter plots with their calculators. We had some technical difficulties in every section getting the applet to work on the first day, but all students were eventually able to use the applet. After completing Parts A and B, we put the activity aside and returned to the textbook to review and extend student knowledge about exponential and logarithmic functions. While students worked on Parts A and B in class, they had review handouts for homework. Kim McDowell created the first one, and I created the second one. Both answer keys are my creations. Most students found these to be reasonable review questions.

Part C of the activity asks students to hypothesize about what type of curve might best fit the data gathered earlier. We talk about how it is easy to determine if data is approximately linear but hard to determine "how curved" nonlinear data is. Although it is not written into the activity, I spend time in class talking with my students about median-median lines and least-squares regression before moving on to the data analysis for the activity. We also talk about how to use residual plots to help determine whether a particular function fits the data well. I plan to incorporate these discussions more formally into the activity, as exposure to statistical ideas is important at the high school level. In the activity, students are asked to use the properties of exponents and logarithms to transform an exponential function into a linear one, and then to do the same with a power function. The transformation of data helps students understand how their calculator performs power and exponential regressions and how it calculates r^2 values. Students also have to transform their linear functions back into exponential or power functions. Transformation of data is a challenging topic for these students, as it is for AP Statistics students. I would like to build up the student work on transformation of data with at least one more example.

I would like to build extensions onto this activity also, looking at some applications of log-log and semi-log plots. I would also like to add a more robust look at creating fractals, either the Koch snowflake or hat fractal or Sierpinski triangles or carpets. With these extensions, I could introduce the idea of self-similarity, and students could focus more on both how the iterated rule generates the design and how the fractal dimension is created. These are all weak spots in the introduction of fractals in the current activity.

5. Probability (Chapter 14 and handouts)

HANDOUTS: INTRODUCTION, PROBABILITY WITH AND/OR, WRAP-UP

ACTIVITY: CONDITIONAL PROBABILITY AND MEDICAL TESTING (Appendix E)

The ideas in probability that are new to students in this course are conditional probability, independence, and expected value. I wrote the introduction handout to give a preview of the unit and encourage students to attend to details. The wrap-up handout, which I also wrote this year, introduces natural frequency as a way to answer conditional probability questions. The handout on probabilities with and/or is primarily Marta Eso's creation, though I did some minor editing for this year. This handout uses tables and helps students build definitions of independence. While it does a nice job presenting both tables and Venn diagrams, it does not go into depth with solving problems using Venn diagrams. A possible extension is the inclusion of a Venn diagram problem with 3 overlapping categories to expand the addition principle (inclusion/exclusion principle). Another possible addition or extension is the tree approach to solving probability problems. The tree approach is the typical way of presenting Bayes' Theorem, and it allows students to visualize a problem that contains only probabilities and nothing about the population

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size in question. None of the handouts currently discuss Bayes' Theorem directly, which is something I would like to address in the future.

The medical testing activity was collaboratively generated in the last couple years. I took earlier versions and significantly rewrote the activity this year to emphasize spreadsheet skills and tie the use of formulas in the spreadsheet directly into the creation of the theoretical calculation of the probability of false positives. I also added the hand calculation of minimum population incidence so that students could see a sample application of a rational inequality. My students are finishing the activity as I finish writing this thesis, so I have not been able to solicit their feedback yet for inclusion in this document.

We will end the fall semester probability unit with a discussion of expected value. In some years, I have groups calculate the size of the jackpot that makes the expected value for a lottery (Powerball or MegaMillions) equal to 0, but I do not expect to have time for that this year.

I have fewer multi-day explorations for the second semester topics, but there are still a number of activities, and I have ideas that may turn into viable explorations.

6. Probability Continued

ACTIVITY: GEOMETRIC DISTRIBUTION PROJECT (Appendix E)

This activity is one that Matthias Weiss and I wrote several years ago. While many precalculus textbooks include questions about binomial distributions, very few include other distributions. We felt that the geometric distribution was an accessible extension to the binomial distribution. As an extension to this activity, I would like to also include some other distributions, even if only in an introductory manner.

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7. <u>Trigonometry</u> (Chapters 5-7)

ACTIVITIES: PREDATOR/PREY and MOVIE SEATING Discovery Projects (Appendix F)

The predator/prey and movie seating activities come from the Discovery Projects section of the textbook companion website, www.stewartmath.com. The predator/prey activity is a straightforward collection of sinusoidal modeling problems, coupled with the addition of sinusoidal equations to form another sinusoid. It also provides the opportunity to introduce students to parametric equations and a phase-plane diagram. I have not written this up as an extension, but it is something that I present in my classroom. I could easily see turning this into an extension.

The Movie Seating activity gets at optimal viewing angles by way of the trigonometric addition and subtraction formulas. Students identify the maximum using features of their graphing calculators, an extension of a technology skill first introduced during our work on polynomial graphs. This is another opportunity to preview calculus ideas. Later in the activity, when the rows of seats are angled, students must use the Law of Cosines to set up their solution.

I have, in the past, done an activity based on Dan Butler's Chocolate Factory Lab that leads students to discover the Law of Cosines using right triangle trigonometry. I have found, however, that the time it takes students to wade through the algebra and symbolic manipulation to derive the formula is not worth the payoff for most of my students; they are able to remember the Law of Cosines without having derived it themselves. While I appreciate proofs and derivations that are accessible to students, I find we spend more time than I would like on trigonometry already. Consequently, I have left this activity out of the course, though it is included in the appendix as a reference.

8. <u>Polar Coordinates, Complex Numbers, and Parametric Equations</u> (Chapter 8)

ACTIVITY: ROADRUNNER AND COYOTE, possibly LINES 3 WAYS (Appendix G)

This Roadrunner and Coyote activity builds on the textbook's discussion of projectile motion as an application of parametric equations. I borrow heavily in this activity from Dan Butler; he showed me his version of the lab several years ago at a conference, using the TI-89. Because the TI-Nspire calculators and computer software now plot entire parametric curves at once and cannot simultaneously trace all three objects (or even two), they are not the best technological choice for this activity. Instead of using this familiar technology, I created a GeoGebra file so that my students can see the paths of the roadrunner, the coyote, and the projectile develop over time. I have not otherwise used GeoGebra with them in this course, nor have they used it in previous courses at our school (we use Geometer's Sketchpad in our Geometry classes), but if they read the handout carefully, they have no difficulty manipulating the GeoGebra file I send them. I have included a screen shot of the file in the appendix.

Having spent significant time with iteration and fractal ideas in the first semester, I would like to develop a programming activity that lets students see Mandelbrot sets for a variety of complex number inputs. Unfortunately, the Discovery Project on fractals on the textbook website gives a TI-83 program, and the TI-Nspires that we use have a different programming structure, so transforming this project into an appropriate activity for my students will require additional work.

The Lines 3 Ways activity is one I created to introduce parametric equations of lines ahead of further parametric work. It assumes knowledge of vectors, so I would need to switch the order in which we cover chapters 8 and 9 in the Stewart text to use this activity. It may make sense to do

so, as this will allow a better understanding of parametric equations before we get to conic sections.

9. <u>Vectors</u> (9.1 and 9.2)

We focus on vectors in only 2 dimensions, so the vast majority of the exercises are application-based. With this in mind, developing vector activities has not been a priority. I would like to include more work in the third dimension. I would also like to include projection vectors. In addition to the physics concept of work, projection vectors present another opportunity to get "under the hood" of the calculator. Least-squares regression is actually a projection vector process that students at this level can understand. An activity here could then lead into working with matrices to solve much larger systems of linear equations and unknowns.

10. <u>Conic Sections</u> (11.1, 11.2, 11.3, 11.4)

ACTIVITY: FERRIS WHEEL (Appendix H)

We did not use the Ferris wheel activity last year, but I plan to use it with my students this year. The activity develops an understanding of the conic sections by parametric definition first, rather than geometric. Conversion of the equations to Cartesian form is done using the Pythagorean Identities. This activity worked well with the Foerster textbook, which builds in more parametric work throughout the year and in the chapter on conic sections. I found that my students came away with a better understanding of the conic sections when I taught them using the parametric definitions first than they typically do when I teach using the geometric definitions first. The Stewart textbook does not have the same support for parametric equations, so we chose to stick with the more traditional geometric approach last year.

11. <u>Limits and Continuity</u> (13.1, 13.2, 13.4)

This has been a short unit in the past, with no time left for an activity.

12. <u>Matrices</u>

ACTIVITY: WATERSHED LAB (Appendix I)

It has been several years since we have included matrices in our precalculus course. I think this is unfortunate, since there are many applications of matrices that might appeal to students while exposing them to significant mathematical concepts. Too often, the title of the course "Precalculus" leads teachers to focus only on mathematical skills directly needed for calculus, at the expense of other rich topics that can enhance mathematical understanding. Matrix algebra, as an example, can serve as a basis to examine the rules and properties of algebra applied to real numbers.

The watershed activity introduces students to transition matrices. While much of the matrix work done at the high school level focuses on using matrices to solve systems of equations, this extension of that idea, iterating a set of rules for a system, would bring nice closure to a year started with the dynamical systems activity. In both cases, we are interested in the long-term behavior of a system defined by applying a simple set of rules to an initial condition. As a possible extension, there are current scientific papers using Leslie matrices (a special form of a transition matrix) to model biological populations that are accessible to students at this level. It would be a valuable experience for students to read one of these articles.

V. DISCUSSION/CONCLUSIONS

There are many challenges to creating and teaching with these activities, but it is worth working to overcome these challenges. Because many of the topics contained in the activities are not part of the typical high school curriculum, they were also new to some of my colleagues. The challenge of educating teachers about the topics/applications covered in these activities might be a significant impediment to their use. While I believe that it is healthy to model some discomfort and discovery with my students, that feeling is not universally shared among teachers (nor among administrators). Equally important, I hope that we can model trying something new and adjusting if it is not going the way we planned. Very seldom is someone's first attempt at any activity, math or otherwise, wildly successful. Discomfort with either the content or the technological tools might be enough to prevent many teachers from using some of these activities. Discomfort with both will almost surely keep many teachers from choosing to use these activities in their classrooms. Building teacher materials and a community of supportive colleagues will go a long way to mitigating the aversion to try something new and daunting. I have work to do to get these materials created, and I challenge our teacher education systems and our schools to foster collaborative environments where we can model lifelong learning for our students.

Another challenge to implementation is the time that it takes for students to work through these activities. The trade-off here is between content coverage (less time focused on content when using the activities) and process and long-term retention. Many of the activities have only been used this year, so there has been no opportunity to learn about long-term retention. As with any educational intervention, the design of a rigorous study of outcomes would be very challenging, particularly given the sample sizes of only one small school (90 precalculus students at our school this year). In the dynamical systems activity, students were at a distinct advantage

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if they could recall work done on iterating linear systems at the start of their Advanced Algebra course the previous year. I was impressed by how much they remembered from that activity, which I will take as a positive sign moving forward.

Part of the challenge, for me, in creating these activities, is to stick within content that is generally considered important for the "standard" student. I get excited about the ideas and extensions, but I have written the activities with an eye towards not taking too much time away from typical high school material. This means that I have glossed over some important background and context. For example, the student handout for the dynamical systems activity does not explain how the linear decrease in growth rate turns into the quadratic form that is iterated.

A major challenge in creating these activities lies in striking the right balance between providing structured support for the students and giving them the opportunity to think independently and engage with the problems. This challenge is made even tougher by the fact that every group of students is different, and the needs of one group may not be the same as the needs of another group. In this way, finding the right balance with these activities is a microcosm of the challenge of finding the right level at which to teach any course. The more feedback I can get from students and colleagues, the better future editions will be at finding this sweet spot.

Even with these challenges, I believe that it is worthwhile to engage teachers and students with this type of activity. In the classroom, it is not enough to focus solely on the goal of having students build mathematical skills and understand mathematical structure. If we want to inspire students to see the beauty and power of mathematics, we need to try to show them how math can be relevant to things they are interested in. We need to engage them in ways that will enhance their long-term retention of ideas, so that we can cut down on re-teaching as topics spiral through

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the curriculum. One way to do this for students is to create activities and a classroom environment in which the mathematics is memorable, meaningful, and fun.

Only a small proportion of students major in mathematics in college. Very few adults use specific mathematical content in their daily lives. Students might, however, use mathematical habits of thinking and a structured approach to problem solving in many facets of their further study and their lives. If you ask a college student about her high school English classes, you are likely to hear about the books she read, the types of writing she did, and the discussions she had in class. If you ask a college student about his high school history classes, you are likely to hear about the time periods and locations he studied, perhaps some surprising details, and maybe about the discussions and presentations in class. If you ask a college student about her high school math classes, on the other hand, you are likely to get a collection of course titles: "oh, I took Algebra and then Geometry, which was pretty cool, and then Algebra 2, and trig was hard." If students have been through an activity-driven curriculum, however, they are more likely to talk about how the iterated quadratic blew up and stopped following a pattern, how to help doctors sort out the likely consequences of diagnostic testing results, or how looking at the same situation from a different perspective shed light on what was happening (cobweb diagrams and phase planes). The activities give context to the mathematics and make it memorable and meaningful.

From my own experiences in the classroom, I can tell you that I certainly have more fun teaching with activities. In terms of modeling for students what it is to be curious, to be a lifelong learner, and to take joy in playing with ideas and solving problems, using activities keeps me motivated and interested. The payoff of enjoying my job more would be big enough for me to continue teaching with activities even if students did not also find the use of activities more fun

than marching through a textbook. Most students do enjoy the activities more than rote symbolic

manipulation, which makes teaching with the activities even better.

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http://www.computerbasedmath.org

APPENDIX A: Unit 1 Handouts and Solutions

MA451 Mathematical Habits and Thinking Handout #1

Here are some rules for thinking, courtesy of Ed Burger and Michael Starbird:
1. Just do it.
2. Make mistakes and fail, but never give up.
3. Keep an open mind.
4. Explore the consequences of new ideas.
5. Seek the essential.
6. Understand the issue.
7. Understand simple things deeply.
8. Break a difficult problem into easier ones.
9. Examine issues from several points of view.
10. Look for patterns and similarities.

(*The Heart of Mathematics; An invitation to effective thinking*, 2000).

Your teacher will take you through the example on prime numbers shown here:
http://www.stewartmath.com/fps/fps-1.pdf

Taking cases can be used both mathematically and with logic, as in the approach to the following problem:
There are two boxes: one marked A and one marked B. Each box contains either a million dollars or a deadly snake that will kill you instantly. You must open one box. On box A there is a sign that reads: "At least one of these boxes contains one million dollars." On box B there is a sign that reads: "A deadly snake that will kill you is in box A." You are told that either both signs are true or both are false. Which box do you open? Be careful!

Once you have been assigned a partner, please work through #1, 11, 12, 14 (skip the distributive property in part c) from this website with your partner. Try to write down your approach, so that you remember how you came up with your solution tomorrow.

For homework, please complete the skills practice checks on Fractions and Equations/Inequalities to begin refreshing your algebra skills.

1. The prime number example is covered on the website.

2. Snakes in boxes: take this one in cases.

       Case 1: The sign on box A is true

This means that at least one box does not have a snake. It also means that the sign on box B is also true, and in this case, we'd want to open B.

       Case 2: The sign on box A is false

This means that neither box contains a million dollars, therefore both boxes contain snakes. If both boxes contain snakes, the sign on box B is true, violating the rules of the game.

Thus, we must be in Case 1, and you should open box B to collect $1 Million.

FOCUS ON PROBLEM SOLVING

1. Is $n^3 - 1$ prime? Students may need to be reminded how to factor a difference of cubes (a good opportunity to have them look inside the front cover of their textbook, which has lots of useful formulas). $n^3 - 1 = (n - 1)(n^2 + n + 1)$ To be prime, one of these factors must equal 1.

       Case 1: $n - 1 = 1$. This yields $n = 2$ and $n^3 - 1 = 7$, which is prime.

       Case 2: $n^2 + n + 1 = 1$. This yields $n = 0$ or $n = -1$, neither of which returns a prime value for $n^3 - 1$.

We can thus conclude that n=2 is the only natural number for which $n^3 - 1$ is prime.

11. George completes a lap in 50s, Sue completes hers in 30s. Clearly, they will both be back at the start after 150s, the LCM of 50 and 30. But can it happen sooner on another portion of the track? We have a feeling it should, since Sue will complete 2 more laps than George in those 150s. Halfway to the 150s mark, at 75s, George will have completed 1.5 laps and Sue will have completed 2.5, so both are halfway around from where they started.

12. Area = ab, Perimeter = 2a + 2b. Set them equal to each other, and then decide which variable to solve for. Since the solutions for one variable in terms of the other are symmetric, it won't matter which a student chooses.

$ab = 2a + 2b$

$ab - 2a = 2b$

$a(b - 2) = 2b$ and we need integer values, so $b > 2$. By symmetry, $b = \dfrac{2a}{a - 2}$ and $a > 2$.

$a = \dfrac{2b}{b - 2}$

Because a and b are interchangeable for height and width, we make a table of values.

a	b
3	6
4	4
5	$\dfrac{10}{3}$, not an integer
6	3

At this point, we've exhausted our symmetric options. The only possible rectangles are 4x4 and 3x6.

14. a) (Baby + Uncle) + Mother = Uncle + Mother = Uncle

b) Father * (Grandpa + Aunt) = Father * Grandpa = Father

c) <u>Commutative</u>: $\begin{matrix} a+b=b+a \\ a*b=b*a \end{matrix}$ The older (younger) of a pair is the same no matter which you list first, so this holds.

<u>Associative</u>: $\begin{matrix} (a+b)+c=a+(b+c) \\ (a*b)*c=a*(b*c) \end{matrix}$ If we take the older (younger) of a and b, and then the older (younger) of that person and c, we will have found the oldest (youngest) of the three people, no matter what order we choose for the pairwise comparisons.

<u>Distributive</u>: Skip this proof for now to avoid confusion. Students get the same effect in part (d) of the question.

d) Let's consider the 2 approaches: (i) $x+(y*z)$, which means to first take the younger of y and z, then to compare this person with x and select the older of the 2; and (ii) $(x+y)*(x+z)$, which means to select the older of x and y and the older of x and z, and then take the younger of the resulting pair. We need to consider 3 cases to see if this holds:

    <u>Case 1</u>: x<y<z. In this case, (i) yields x+y=y and (ii) yields y*z=y, so the result is the same.

    <u>Case 2</u>: y<x<z. (i) gives x+y=x and (ii) gives x*z=x, so the result is again the same.

    <u>Case 3</u>: y<z<x. (i) give x+y=x and (ii) gives x*x=x.

In all three cases, the results are the same. The real numbers, however, do not share this property.

$$1+(2*3)\neq(1+2)*(1+3)$$

For example, $1+6\neq 3*4$

$$7\neq 12$$

Fractions Skills Test
Practice Test #1

Answer each question simplifying as much as possible.
Express your answer as one reduced fraction.

1. $\dfrac{3}{5} - \dfrac{4}{3}$

2. $\dfrac{x}{yz} + \dfrac{y}{z}$

3. $\dfrac{3x}{5y} \div \dfrac{4x}{2y^2}$

4. $\dfrac{3}{5} \bullet \dfrac{4}{3} \bullet \dfrac{5}{2}$

5. $\dfrac{3}{4(x+1)} - \dfrac{7}{2(x-1)}$

6. $\dfrac{\dfrac{x^2-4}{x+1}}{\dfrac{x+2}{3x-5}}$

7. $\dfrac{xy}{yz} - \dfrac{y}{z}$

8. $\dfrac{A}{x-1} + \dfrac{B}{x+2}$

9. $\dfrac{1 + \dfrac{3}{2}}{\dfrac{3}{4} - 1}$

10. $\dfrac{x + \dfrac{y}{z}}{\dfrac{y}{z} - z}$

1._____

2._____

3._____

4._____

5._____

6._____

7._____

8._____

9._____

10._____

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MA 350 Skills Practice

Equations and Inequalities Skills Check
Practice #1

1. Solve the given equation for x. $7 - 3x = -14$

1. _____

2. Solve the given equation for y. $\dfrac{2(y-1)}{3} = 4$

2. _____

3. Solve the given equation for t. $-2(t-1) = 1 + t$

3. _____

4. Solve the given equation for n. $5 - 4(n+1) = 12n - 31$

4. _____

5. Write the solution to the given inequality in interval notation. $4 - 3x > 2$

5. _____

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6. Solve the given equation for x. $\dfrac{10}{x} = \dfrac{6}{x+2}$

7. Solve the given equation for z. $-3z - 10y = 13x$

8. Solve the given equation for h. $A = \dfrac{1}{2}(b+c) \cdot h$

9. Write the solution to the given inequality in interval notation. $|x-4| > 1$

10. Solve the system of two equations and two unknowns.
$3x - 4y = 10$
$5x + 8y = 2$

6. _____

7. _____

8. _____

9. _____

10. $x = $ _____ $y = $ _____

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MA451 Mathematical Habits and Thinking Handout #2

It's always good to think about multiple ways that you could approach a problem. Yesterday's class included some insights on splitting a problem to look at cases. Hopefully your teacher has helped you remember two approaches to dealing with compound fractions and also reminded you of factoring techniques like splitting the middle term, in addition to trial and error. Another tremendously helpful mathematical skill is recognizing patterns. In factoring, some common patterns are differences of squares and perfect square trinomials. For other questions, it may not be so obvious what pattern will emerge. If a question is asking you about a repeated process (composition of functions, exponentiation, etc.), then trying to find a pattern will save you a LOT of work.

Your teacher will now take you through the examples from:
http://www.stewartmath.com/fps/fps-2.pdf

When you have a new partner, please work with that person on # 1, 3, 5 from the handout. The more you can write down about your process in problem-solving, the better.

For homework, continue to refresh your algebra skills by completing the skills practice handouts on Functions and Toolkit Functions. Algebra 2 Skills Checks Resources can be found at the Math Departments internal website:
https://sites.google.com/a/hotchkiss.org/ma350/resources/ma350-skills-test-resources

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1. The calculator value for $947^{362} = \infty$, so we need a new tactic. Try making a table:

Power	Final Digit
947^1	7
947^2	9
947^3	3
947^4	1
947^5	7
947^6	9
947^7	3

And we see that the final digit repeats every 4 powers. 362 divided by 4 is 90 with a remainder of 2, so the final digit is the 2nd value in the cycle, 9.

3. $f_0(x) = \dfrac{1}{2-x}$, and $f_{n+1}(x) = f_0(f_n(x))$ for n=0, 1, 2, …

$$f_1(x) = f_0(f_0(x)) = \frac{1}{2 - \dfrac{1}{2-x}} \cdot \frac{2-x}{2-x} = \frac{2-x}{4-2x-1} = \frac{2-x}{3-2x}$$

$$f_2(x) = f_0(f_1(x)) = \frac{1}{2 - \dfrac{2-x}{3-2x}} \cdot \frac{3-2x}{3-2x} = \frac{3-2x}{6-4x-2+x} = \frac{3-2x}{4-3x}$$

$$f_3(x) = f_0(f_2(x)) = \frac{1}{2 - \dfrac{3-2x}{4-3x}} \cdot \frac{4-3x}{4-3x} = \frac{4-3x}{8-6x-3+2x} = \frac{4-3x}{5-4x}$$

We see the pattern now, that $f_n(x) = \dfrac{(n+1) - nx}{(n+2) - (n+1)x}$, and thus $f_{100}(3) = \dfrac{101 - 100 \cdot 3}{102 - 101 \cdot 3} = \dfrac{199}{201}$.
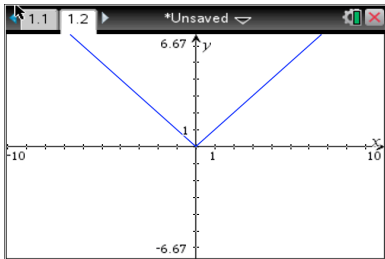
5. $\sqrt{1 - \sqrt{2 - \sqrt{3-x}}}$ Start inside the most restricted root and then move out. Because $3 - x \geq 0$, we have $x \leq 3$. Moving out, $\sqrt{3-x} \leq 2 \Rightarrow 3-x \leq 4 \Rightarrow -1 \leq x$. Finally, we turn to the outer root $\sqrt{2 - \sqrt{3-x}} \leq 1 \Rightarrow 2 - \sqrt{3-x} \leq 1 \Rightarrow \sqrt{3-x} \geq 1 \Rightarrow 3-x \geq 1 \Leftarrow x \leq 2$. Putting it all together, we have that $-1 \leq x \leq 2$.

The Hotchkiss School   Name: _____
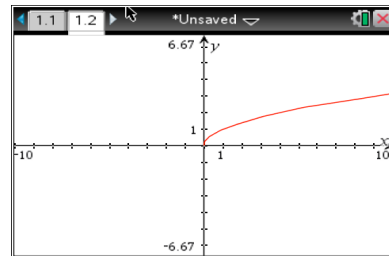MA 350 Skills Practice Check

Tool Kit Functions Skills Check

For questions 1 – 5, write the equation for the graph shown

1.
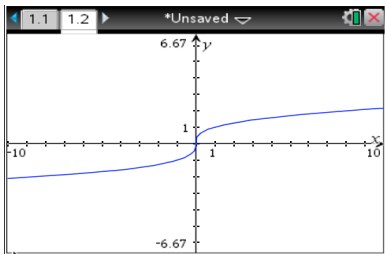

2.
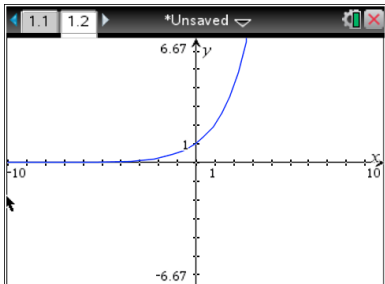

3.

4.

5.

1. _____

2. _____

3. _____

4. _____

5. _____

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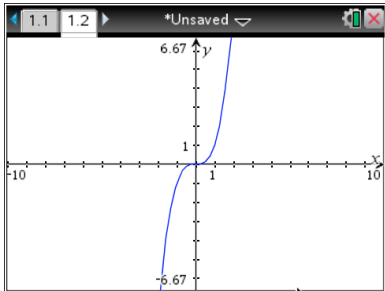
6. State the domain of the function $y = \sqrt[3]{x}$.

7. State the range of the function $y = \log(x)$.

8. Write an equation for the function described. *The square root function shifted left 2 and reflected over the x-axis.*

9. Shown below are a parent function $f(x)$ and a transformation $g(x)$. Write an equation for $g(x)$ in terms of $f(x)$.
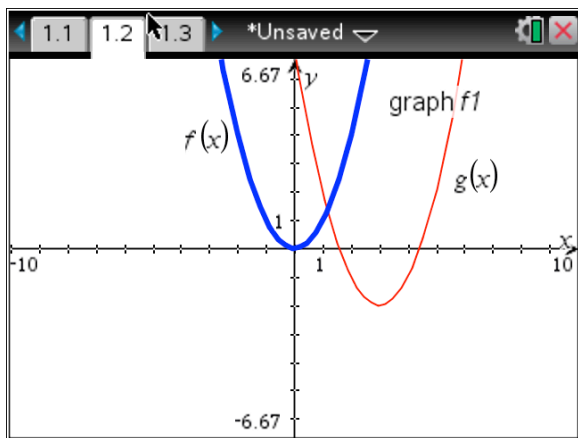
10. If the point (1, -2) lies on the graph of $y = f(x)$, give the coordinates of one point that lies on the graph of $y = 2f(x) - 3$.

6. _____

7. _____

8. _____

9. _____

10. _____

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MA451 Mathematical Habits and Thinking Handout #3

By now you've reviewed some algebra, you've thought more about problem-solving strategies, and you're getting back in the habit of talking about mathematical ideas with your peers. This is a great start. Sometimes, though, a problem requires some common sense and the application of non-mathematical knowledge. You should know, too, that mathematicians prize elegant solutions, those that limit brute force by finding efficient approaches.

1. Two rooms are connected by a hallway that has a bend in it, so that it is impossible to see one room while standing in the other. One of the rooms has three light switches. You are told that exactly one of the switches turns on a light in the other room, and the other two are not connected to any lights. What is the fewest number of times you would have to walk to the other room to figure out which switch turns on the light?

Why is the answer to the preceding question "one"? Would you choose this approach if your goal was something other than minimizing the number of trips down the hallway?

2. A passenger train traveling at a steady 50 miles per hour left Austin, Texas at noon bound for Dallas, exactly 210 miles away. At the same instant, a freight train traveling at a steady 20 miles per hour left Dallas headed for Austin on the same track. At this same high noon, a fly leaped from the nose of the passenger train and flew along the track at 100 miles per hour. When the fly touched the nose of the oncoming freight train, she turned and flew back toward the passenger train at 100 miles per hour. When she reached the nose of the passenger train, she instantly turned again and started flying back toward the freight train. She continued flying in this pattern until the expected tragedy occurred—she was squashed as the trains collided head on. Figure out how far the fly had flown before her untimely demise. (Hint: you can get there with brute force, but there's a more elegant way.)

For homework, please complete the algebra skills practice handout on Toolkit Functions. Then, please reflect on, and write answers to these questions:
What makes a good math student? What makes a good math teacher?

Your answers to these questions will be collected and summarized to create a document reflecting the input of the whole class.

1. If you turn on a light bulb, it warms up. Since the goal is to minimize the number of trips down the hallway, not to minimize the time it takes to answer the question, you should flip switch 1 to ON, wait a few minutes, then switch it back to OFF. You should then immediately flip switch 2 to ON and walk down the hallway. If the light is on, switch 2 controls it. If the light is off but warm, then switch 1 controls it. If the light is off and room temperature, then switch 3 controls it.

2. Students may be tempted to work this problem by force, calculating the times and distances for each change of direction for the fly. This method *can* yield the right answer (300 miles). A more elegant solution is to answer a related question: "how long until the trains collide?" We know d=rt, and the fly is moving at 100 mph. If we know the time until the collision, calculating the distance flown is trivial. The gap between the trains is shrinking at 50 + 20 = 70 mph, and they start 210 miles apart. Therefore the trains will collide in 3 hours (at 3PM), and we know the fly will have flown 300 miles.

Name: _____

Functions and Function Notation Skills Check
Practice Check #1

1. Consider the function $f(x) = 2x^2 - 3x$. Evaluate $f(-2)$.

2. Determine the domain of the function $f(x) = \dfrac{1-x}{x^2}$.

3. Let $f(x) = x + 2$ and $g(x) = 1 - x^2$. Compute and simplify $g(f(x))$.

4. Consider $g(x) = \dfrac{2-x}{1+x^2}$. Evaluate $g(x+h)$.

5. Let $f(x) = 1 + 3x - 2x^3$. Determine which is larger, $f(-1)$ or $f(3)$. If the quantities are equal, write "equal".

6. Determine the range of the function $g(x) = (x-1)^2 + 3$.

7. Determine which of the following relations represent functions. List all that apply.

A

$f(x)$	0	3	6	9	12
x	-2	-1	0	1	2

B

$g(x)$	0	1	0	1	0
x	-2	-1	0	1	2

C

$h(x)$	0	3	6	9	12
x	-2	-1	0	-1	-2

8. Is the function $f(x) = \sqrt{x}$ increasing or decreasing?

9. Determine the x and y intercepts for the function $y = 2x - 6$.

10. Let $f(x) = \dfrac{1+x^2}{3x-5}$. Evaluate and simplify $f\left(\dfrac{1}{y}\right)$.

1. _____

2. _____

3. _____

4. _____

5. _____

6. _____

7. _____

8. _____

9. x-intercept: _____

   y-intercept: _____

10. _____

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APPENDIX B: Unit 2 Activity and Solutions
I. Introduction and Reminders: Tools and Constant Growth Rate
If you were in MA350 last year, you started the year with the drug lab, which we'll refresh now (and introduce to students who haven't seen it) before exploring a more complex model. In the first drug lab model, you ingested a drug, and then your body processed and removed one-quarter of the drug in each 4-hour time step, yielding a *growth rate* of $\frac{3}{4}$. You initially defined the relationship *recursively (sometimes called implicitly)*, with values defined based on the previous value and the initial value needed to determine behavior:

$$\begin{cases} A(0) = 16 \\ A(t+1) = \frac{3}{4}A(t) \end{cases} \quad or \quad \begin{cases} D_0 = 16 \\ D_t = \frac{3}{4}D_{t-1} \end{cases}$$

We'll stick to the function notation shown first, though we could use the sequence notation shown second. You learned how to use your calculator to simplify the calculations, and it wasn't too difficult to derive an *explicit* model, which allowed you to calculate a value at any time step without iterating all the steps prior to that one.

$$A(t) = A(0)\left(\frac{3}{4}\right)^t$$

1. The initial model started with 16ml of drug in the system. How much drug was left after the 10th iteration? How would you have obtained this answer if you only had the recursive model?

You then created a more complicated model by considering what happened if, instead of only a single dose, you took an initial dose of the drug and then a maintenance dose every 4-hour time step. In this case, the recursive model was still fairly easy to generate:

$$\begin{cases} A(0) = 16 \\ A(t+1) = \frac{3}{4}A(t) + 8 \end{cases}$$

It was much harder, however, to come up with an explicit model, either through logic or by looking at the plot of Amount of Drug in the System vs. Time (a time series plot). An explicit model can be obtained algebraically by rewriting the general term in terms of the initial amount and finding the sum of the resulting geometric series. The explicit model turns out to be a transformed exponential function, with the asymptotic behavior explaining the long-term trend towards one value.

It was possible, however, to use a different tool, called a "cobweb" or "input-output" diagram, to help you see what was happening and find the long-term behavior, even without the explicit

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function. As a quick refresher on cobweb diagrams, please go to
http://mathinsight.org/applet/function_iteration_cobweb_combined
First, your teacher will show you how to resize the window so that the y-scale goes up to 40.
Then, enter $f(x) = \frac{3}{4}x$ in the first box, and $x_0 = 16$ in the second box. Press the "iterate" button
repeatedly until you can see what's going on with both diagrams (cobweb on left, time series on
right). Then, hit the "reset" button and change $f(x) = \frac{3}{4}x + 8$, leaving $x_0 = 16$.

2. Can you tell how the cobweb diagram helps you trace the behavior over time? What visual
clues do you get from the cobweb diagram? What role does the line $y = x$ play?

3. What happens if you enter a growth rate larger than 1, rather than ¾? Try $f(x) = 1.07x$ to see
what happens to an initial quantity defined by this recursive equation.

The process of exploring the behavior involved the following pieces:
- a recursive definition;
- a table of values;
- a time series plot to see what was happening over time;
- a cobweb diagram to try to explain long-term behavior;
- an explicit model (often the hardest step).
We'll use these same tools in examining a non-linear iteration model.

II. New Patterns of Behavior: Variable Growth Rate
A. Background

The model used in the drug lab, whether including the maintenance dose or not, assumes a constant rate of growth/decay (in this case $\frac{3}{4}$), no matter what the quantity is. Especially in terms of population growth, this is not always a valid assumption. There are often factors, including declining resource availability, increased predation pressure, and a higher incidence of disease, that slow the rate of growth of a population as the population size increases.

We will now examine a model that assumes *growth rate decreases as population size increases*. We'll use a simple model, in which we assume that the growth rate decreases linearly as the population size increases. While this assumption may seem basic, it has been shown to be present in the growth of yeast in test tubes with nutrients (Mable and Otto, 2001), in many insect populations (Hassell et al, 1976), and perhaps in tumor growth (Cross and Cotton, 1994). Rather than fill in biological background, we'll look at *the spread of a disease among students at Hotchkiss*.

B. The Model

To simplify the equations, calculations, and some of the information needed, we will shift x to represent the **proportion of the population that is currently sick**. In this case, $x = 1$ represents a population with 100% of the population (all 600 students) sick, and $0 < x < 1$ represents a population with both sick and healthy individuals. The proportion of infected students at time $t + 1$ depends on:

- $x(t)$, the proportion of students currently infected;
- $(1 - x(t))$, the proportion of students not infected;
- and r, a growth factor that takes into account both the likelihood of interaction between students and the virulence of the disease.

We'll assume that 24 people (0.04 of the total) are initially sick. The recursive equation, then, is:

$$\begin{cases} x(0) = 0.04 \\ x(t+1) = rx(t)(1 - x(t)) \end{cases}$$

4. To get a sense of how the dynamic plays out, we'll create a table, by hand, with the first 10 iterates (or steps) with 24 people sick initially, so that $x(0) = 0.04$, and a disease with $r = 1.5$. We'll look at other r values later.

Remember that the output from one step becomes the input for the next step.

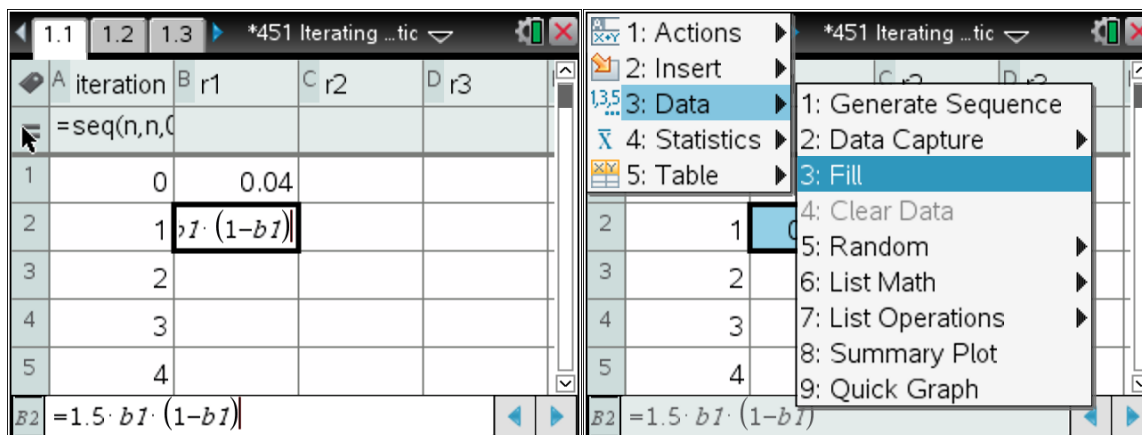| Iteration	Input (Percent Sick)	Output (Percent Sick)
1	0.04	0.0576
2	0.0576	
3		
4		
5		
6		
7		
8		

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9		
10		

5. Plot the ordered pairs of (Iteration, Output) on the attached graph paper.

6. What proportion of the population is sick after the 5th iteration? How many students is this?

C. Technology to Remove Tedium

To see more steps easily, open a Lists and Spreadsheets page on your calculator. Title the first column "Iteration" and enter the formula "$=seq(n,n,0,20)$" in the formula bar. This should fill the first column with the integers from 0 to 20. Title column B "r1" and enter 0.04 in cell B1. Once this is entered, put this formula in cell B2: $=1.5*b1*(1-b1)$. Click on cell B2, then click Menu-->Data-->Fill, and use the arrow keys to highlight down to cell B21.

To create a time-series graph, add a Data and Statistics page to your document. Choose "iteration" as the independent variable and "r1" as the dependent variable.

7. Describe what is happening to the values as you iterate. Are they stabilizing? If so, at what value? If they are not stabilizing at one value, describe any patterns that you see.

8. Now create a cobweb (Input-Output) plot on the attached graph. You will do this twice, with 2 different starting values, 0.04 and 0.5. Please use different colors for each. What do you notice about the graphs?
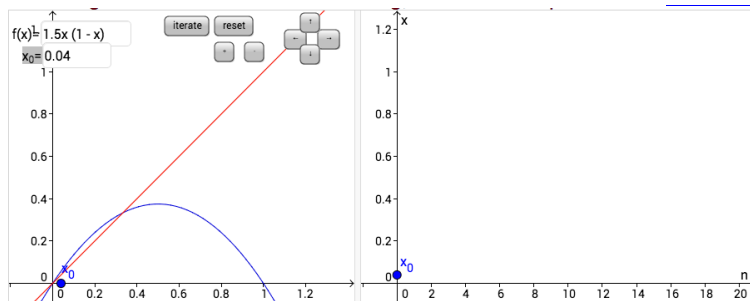
What happens if we have a disease with a higher r value, indicating a greater likelihood of transmission? Let's test this in column C, which you should now title "r2". Again enter the value of 0.04 in cell C1, and the formula $= 2.5 * c1 * (1 - c1)$ in cell C2. Then select the cell and fill down so that you have values for 20 iterations. Add another Data and Statistics page to your document, this time plotting "r2" as the dependent variable.

9. Describe what is happening to the values as you iterate with this higher r value. Are they stabilizing? If so, at what value? If they are not stabilizing at one value, describe any patterns that you see. What similarities do you notice in comparing this time-series plot to the first plot? What is different?

In both cases, the values have stabilized, but to different proportions of the population that are sick. If we have information about the virulence of the disease, a big contributor to the r value, can we predict what that stabilizing value will be? From the information we have so far, it's challenging. Let's return to the cobweb plot to see if that provides any clues. In case you closed the link, here it is again:

http://mathinsight.org/applet/function_iteration_cobweb_combined

This time, resize the windows using the + and the arrow keys so that each axis goes from 0 to 1.2. Then, change the function and x_0 values so that $f(x) = 1.5 * x * (1 - x)$ and $x_0 = 0.04$.

Iterate 20-30 times and and see what happens.

Hit the "reset" button, then change the function so that $f(x) = 2.5 * x * (1 - x)$. Iterate 20-30 times again.

10. What do you notice? Can you make an educated guess about where the values stabilize and how that is represented graphically in the cobweb diagram? Is this the same as for the linear functions you iterated last year? Confirm your guess by solving algebraically for the stabilizing value when $r = 1.5$.

11. Now predict the expected long-term behavior if $r = 3.5$ using the same algebra work as above.

Test your hypothesis by returning to the applet at
http://mathinsight.org/applet/function_iteration_cobweb_combined
12. Plug in $f(x) = 3.5 * x * (1 - x)$ and a seed value of 0.04 for x_0. Describe the pattern that emerges as you iterate. Is it what you expected?

13. Mathematically, this system, as was the linear system before, is a **discrete dynamical system**. Discrete means that it moves in chunks, from time 0 to time 1, rather than continuously along from 0 to 1. Look up "dynamical system" and put the definition into your own words below.

As you can tell from just the 3 examples so far, the behavior of discrete dynamical systems is highly dependent on both initial value and the values of parameters in the equations. It is not possible, even for this relatively simple model, to create an explicit formula that will tell us exactly what proportion of the population will be sick at any given time; we can only march through each iteration one by one. Return to the applet and see if you can find an r value for which $x(t)$ cycles between 2 distinct values in the long term. Once you have one, find a second one. Hint: if you don't press "reset" when you change $f(x)$, the number of iterations from your previous function will remain. Also try to find an r value that produces a system in which no discernible pattern emerges. In our case, we'll see no discernible pattern for the proportion of the population with the disease. To find out more, see this link.

14. The system oscillates between 2 values when $r =$ _____. The values are_____ and _____.

15. The system oscillates between 3 or 4 values when $r =$ _____.

The values are_____and_____and_____and_____.

16. The system does not exhibit any discernible pattern when $r =$ _____or_____

17. Compare the r values you've found with your classmates. Try to spot patterns of long-term behavior based on r value.

D. The Continuous Case
Interestingly, the seeming disorder inherent in this system disappears if we use *continuous* time rather than *discrete* time. In fact, in continuous time, this same initial set up has the **logistic growth** function as its explicit solution. Depending on the application, the logistic function will take on various forms. Your text uses the form $N(t) = \dfrac{c}{a + be^{-rt}}$ where $y = \dfrac{c}{a}$ is the horizontal asymptote of the graph, sometimes called the carrying capacity, and r represents the initial growth rate. We'll see more of this function later in the year. For now, to see how well a logistic curve fits the well-behaved data from functions with low r value, go back to your calculator document and find the page with the time-series graph for $r = 1.5$. Click the "Menu" button, then select "4: Analyze", then "6: Regression" and scroll down to select "B: Logistic Regression (d=0)."
18. Write down the logistic equation that fits your data for $r = 1.5$.

19. Repeat this process and write down the logistic equation that fits your data for $r = 2.5$.

Please note that the tools of modeling you've used here:
- recursive equations,
- tables,
- time-series plots,
- cobweb diagrams, and
- explicit equations when available

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all help you understand the system you're modeling in slightly different ways, and that a progression through multiple tools is often the best way to really understand a system.

(Iteration, Output)
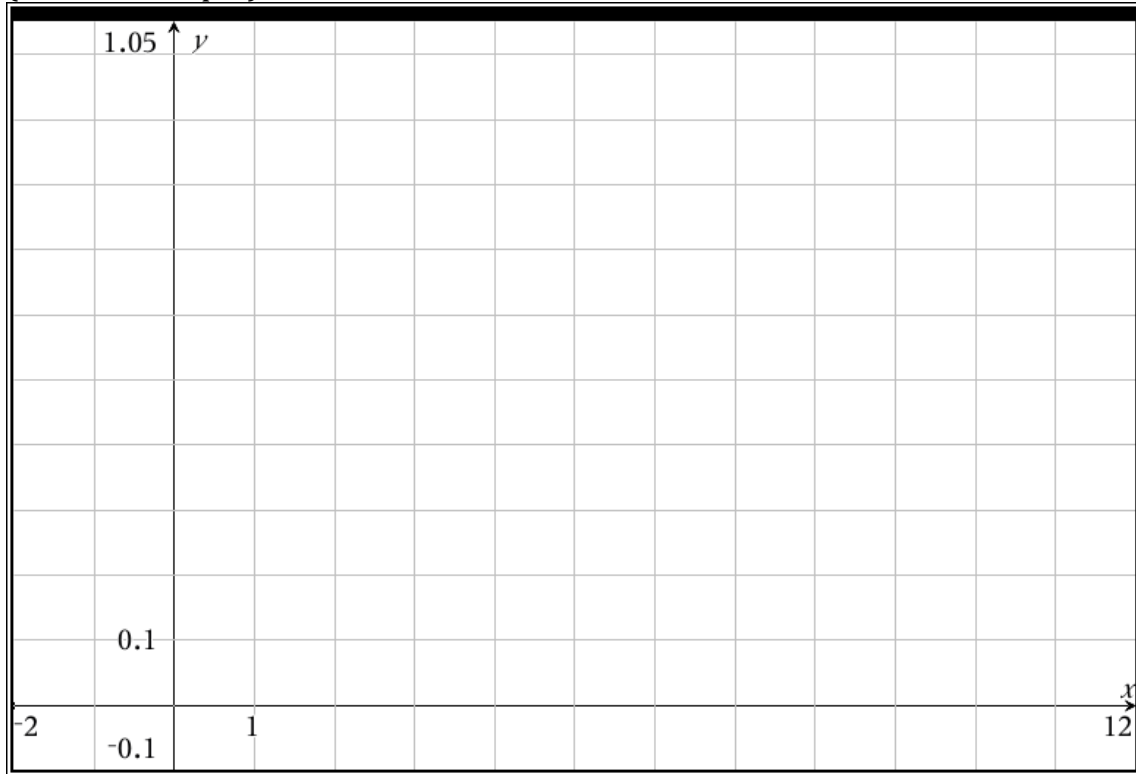
(Input, Output)

I. Introduction and Reminders: Tools and Constant Growth Rate

If you were in MA350 last year, you started the year with the drug lab, which we'll refresh now (and introduce to students who haven't seen it) before exploring a more complex model. In the first drug lab model, you ingested a drug, and then your body processed and removed one-quarter of the drug in each 4-hour time step, yielding a *growth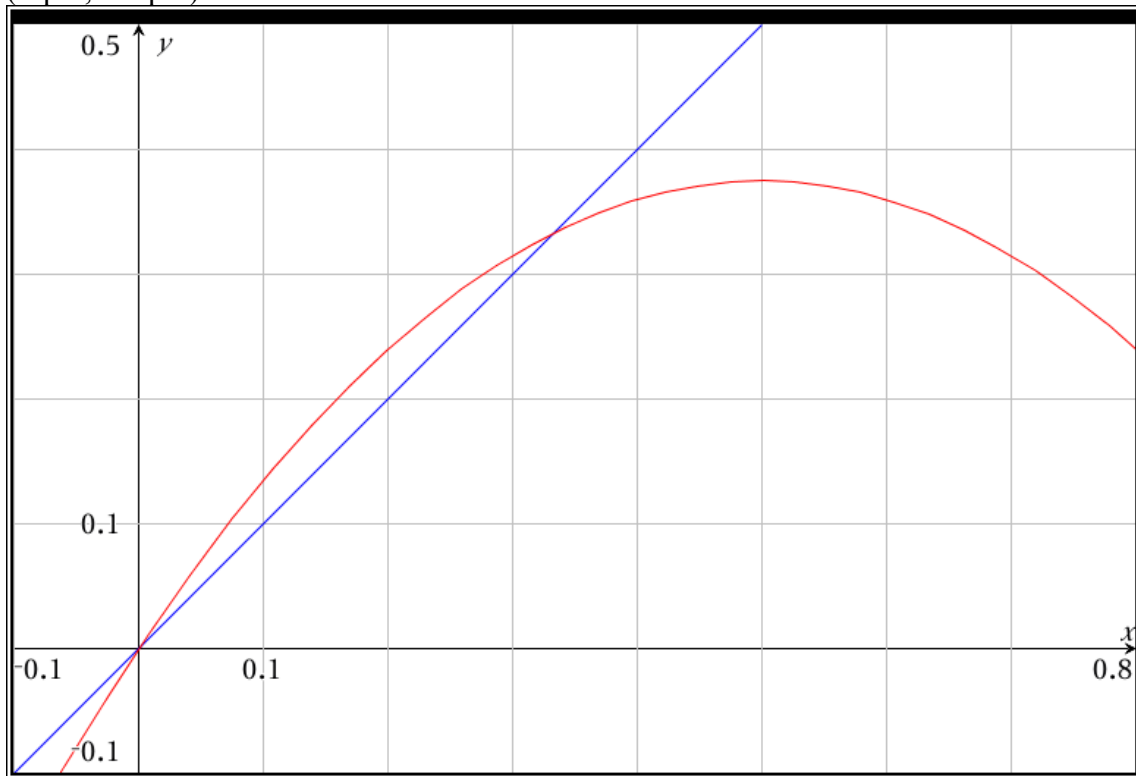 rate* of $\frac{3}{4}$. You initially defined the relationship *recursively (sometimes called implicitly)*, with values defined based on the previous value and the initial value needed to determine behavior:

$$\begin{cases} A(0)=16 \\ A(t+1)=\dfrac{3}{4}A(t) \end{cases} or \begin{cases} D_0 = 16 \\ D_t = \dfrac{3}{4}D_{t-1} \end{cases}$$

We'll stick to the function notation shown first, though we could use the sequence notation shown second. You learned how to use your calculator to simplify the calculations, and it wasn't too difficult to derive an *explicit* model, which allowed you to calculate a value at any time step without iterating all the steps prior to that one.

$$A(t) = A(0)\left(\frac{3}{4}\right)^t$$

1. The initial model started with 16ml of drug in the system. How much drug was left after the 10th iteration? How would you have obtained this answer if you only had the recursive model?

$A(10) = 16\left(\dfrac{3}{4}\right)^{10} = 0.901$ ml of drug after 10 iterations.

If we had only the recursive model, I would have had to hit "enter" 10 times on my calculator to get the value. The explicit model makes it much easier to predict behavior over time.

You then created a more complicated model by considering what happened if, instead of only a single dose, you took an initial dose of the drug and then a maintenance dose every 4-hour time step. In this case, the recursive model was still fairly easy to generate:

$$\begin{cases} A(0)=16 \\ A(t+1)=\dfrac{3}{4}A(t)+8 \end{cases}$$

It was much harder, however, to come up with an explicit model, either through logic or by looking at the plot of Amount of Drug in the System vs. Time (a time series plot). An explicit model can be obtained algebraically by rewriting the general term in terms of the initial amount and finding the sum of the resulting geometric series. The explicit model turns out to be a transformed exponential function, with the asymptotic behavior explaining the long-term trend towards one value.

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repeatedly until you can see what's going on with both diagrams (cobweb on left, time series on

right). Then, hit the "reset" button and change $f(x) = \frac{3}{4}x + 8$, leaving $x_0 = 16$.

2. Can you tell how the cobweb diagram helps you trace the behavior over time? What visual
clues do you get from the cobweb diagram? What role does the line $y = x$ play?
Yes; the amount of drug in the system seems to approach the intersection of the 2 lines, as it did
in the initial does only situation. The line $y = x$ helps by allowing easy transitions as the output
(y) of one iteration becomes the input (x) of the next. It is much easier to find the intersection of
2 lines algebraically than it is to come up with an explicit model for this case so that we could
use that to predict long-term behavior.

3. What happens if you enter a growth rate larger than 1, rather than ¾? Try $f(x) = 1.07x$ to see
what happens to an initial quantity defined by this recursive equation.
The amount grows without bound, gaining more each iteration as the lines grow further apart. In
this case, movement is away from the point of intersection at (0,0), rather than towards it.

The process of exploring the behavior involved the following pieces:
- a recursive definition;
- a table of values;
- a time series plot to see what was happening over time;
- a cobweb diagram to try to explain long-term behavior;
- an explicit model (often the hardest step).
We'll use these same tools in examining a non-linear iteration model.

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A. Background
The model used in the drug lab, whether including the maintenance dose or not, assumes a

constant rate of growth/decay (in this case $\frac{3}{4}$), no matter what the quantity is. Especially in

terms of population growth, this is not always a valid assumption. There are often factors,
including declining resource availability, increased predation pressure, and a higher incidence of
disease, that slow the rate of growth of a population as the population size increases.

We will now examine a model that assumes *growth rate decreases as population size increases*.
We'll use a simple model, in which we assume that the growth rate decreases linearly as the
population size increases. While this assumption may seem basic, it has been shown to be
present in the growth of yeast in test tubes with nutrients (Mable and Otto, 2001), in many insect

populations (Hassell et al, 1976), and perhaps in tumor growth (Cross and Cotton, 1994). Rather than fill in biological background, we'll look at *the spread of a disease among students at Hotchkiss*.

B. The Model

To simplify the equations, calculations, and some of the information needed, we will shift **x** to represent the **proportion of the population that is currently sick**. In this case, $x = 1$ represents a population with 100% of the population (all 600 students) sick, and $0 < x < 1$ represents a population with both sick and healthy individuals. The proportion of infected students at time $t + 1$ depends on:

- $x(t)$, the proportion of students currently infected;
- $(1 - x(t))$, the proportion of students not infected;
- and *r*, a growth factor that takes into account both the likelihood of interaction between students and the virulence of the disease.

We'll assume that 24 people (0.04 of the total) are initially sick. The recursive equation, then, is:

$$\begin{cases} x(0) = 0.04 \\ x(t+1) = rx(t)(1 - x(t)) \end{cases}$$

4. To get a sense of how the dynamic plays out, we'll create a table, by hand, with the first 10 iterates (or steps) with 24 people sick initially, so that $x(0) = 0.04$, and a disease with $r = 1.5$. We'll look at other *r* values later.

Remember that the output from one step becomes the input for the next step.

Iteration	Input (Percent Sick)	Output (Percent Sick)
1	0.04	0.0576
2	0.0576	0.08142
3	0.08142	0.11219
4	0.11219	0.14940
5	0.14940	0.19062
6	0.19062	0.23143
7	0.23143	0.26680
8	0.26680	0.29343
9	0.29343	0.31099
10	0.31099	0.32141

5. Plot the ordered pairs of (Iteration, Output) on the attached graph paper.
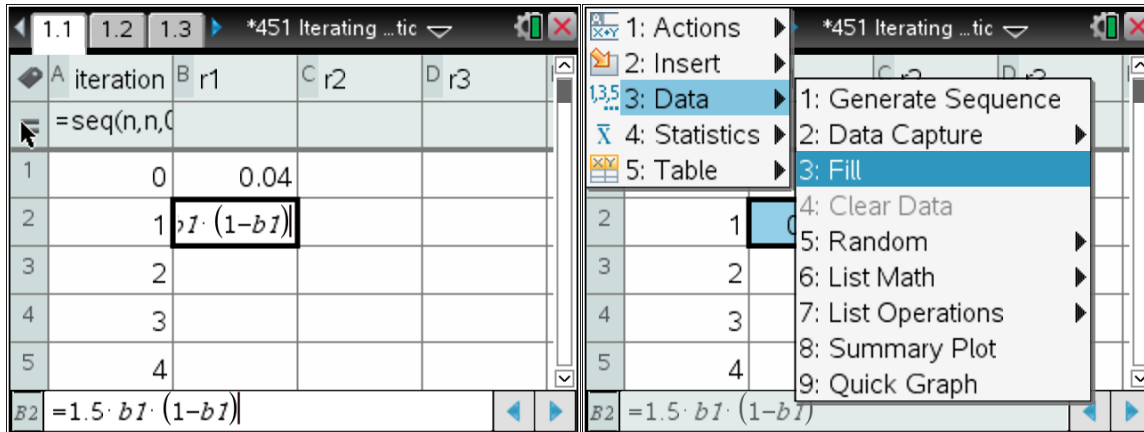This should begin to look like a logistic plot, though it is not quite enough iterations to see the stabilization around 0.333. It is a by-hand creation of the time-series plot.

6. What proportion of the population is sick after the 5th iteration? How many students is this?

$0.190625313 * 600 = 114.375$, so 114 students are sick.

C. Technology to Remove Tedium

To see more steps easily, open a Lists and Spreadsheets page on your calculator. Title the first column "Iteration" and enter the formula "=seq(n,n,0,20)" in the formula bar. This should fill the first column with the integers from 0 to 20. Title column B "r1" and enter 0.04 in cell B1. Once this is entered, put this formula in cell B2: $=1.5*b1*(1-b1)$. Click on cell B2, then click Menu-->Data-->Fill, and use the arrow keys to highlight down to cell B21.

To create a time-series graph, add a Data and Statistics page to your document. Choose "iteration" as the independent variable and "r1" as the dependent variable.

7. Describe what is happening to the values as you iterate. Are they stabilizing? If so, at what value? If they are not stabilizing at one value, describe any patterns that you see.

The initial growth is concave up, but it then slows and the values appear to be stabilizing at around 1/3 of the population (200 students) getting sick.

8. Now create a cobweb (Input-Output) plot on the attached graph. You will do this twice, with 2 different starting values, 0.04 and 0.5. Please use different colors for each. What do you notice about the graphs?

Both graphs move towards the intersection point that is not (0,0). It may surprise students that starting with more than the equilibrium proportion will still yield the same equilibrium value.

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What happens if we have a disease with a higher *r* value, indicating a greater likelihood of transmission? Let's test this in column C, which you should now title "r2". Again enter the value of 0.04 in cell C1, and the formula $= 2.5 * c1 * (1 - c1)$ in cell C2. Then select the cell and fill down so that you have values for 20 iterations. Add another Data and Statistics page to your document, this time plotting "r2" as the dependent variable.

9. Describe what is happening to the values as you iterate with this higher *r* value. Are they stabilizing? If so, at what value? If they are not stabilizing at one value, describe any patterns that you see. What similarities do you notice in comparing this time-series plot to the first plot? What is different?

The values grow more quickly and larger in this scenario. They stabilize around 0.6, but, unlike in the previous plot, the values went above that threshold value briefly (4, 0.6108) before settling back in. In both cases, growth started concave up and then stabilized near a particular proportion.

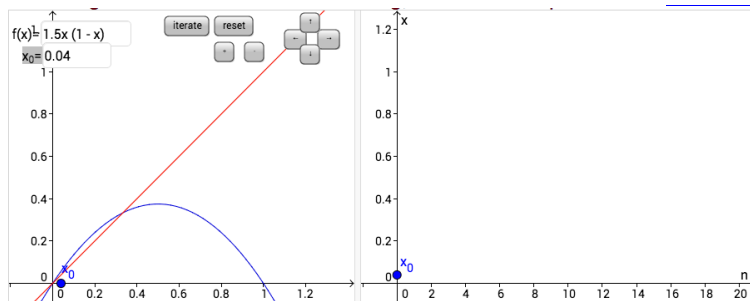In both cases, the values have stabilized, but to different proportions of the population that are sick. If we have information about the virulence of the disease, a big contributor to the r value, can we predict what that stabilizing value will be? From the information we have so far, it's challenging. Let's return to the cobweb plot to see if that provides any clues. In case you closed the link, here it is again:

http://mathinsight.org/applet/function_iteration_cobweb_combined

This time, resize the windows using the + and the arrow keys so that each axis goes from 0 to 1.2. Then, change the function and x_0 values so that $f(x) = 1.5 * x * (1 - x)$ and $x_0 = 0.04$.

 Iterate 20-30 times and and see what

happens.

Hit the "reset" button, then change the function so that $f(x) = 2.5 * x * (1 - x)$. Iterate 20-30 times again.

10. What do you notice? Can you make an educated guess about where the values stabilize and how that is represented graphically in the cobweb diagram? Is this the same as for the linear functions you iterated last year? Confirm your guess by solving algebraically for the stabilizing value when $r = 1.5$.

The values in both cases approach the nonzero intersection between the parabola and the line $y = x$, though the path of the cobweb for $r = 2.5$ leaves the region between the 2 curves. This is the first time students have seen a cobweb path that leaves the space between curves.

Some students forget their algebra and struggle with solving the quadratic that results from setting the 2 functions equal to solve for the intersection.

$$x = 1.5x(1 - x)$$
$$x = 1.5x - 1.5x^2$$
$$0 = 0.5x - 1.5x^2$$
$$0 = x(0.5 - 1.5x)$$
$$x = 0 \_ or \_ 0.5 - 1.5x = 0$$
$$0.5 = 1.5x$$
$$x = \frac{0.5}{1.5} = \frac{1}{3}$$

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11. Now predict the expected long-term behavior if $r = 3.5$ using the same algebra work as above.

$$x = 3.5x(1-x)$$
$$x = 3.5x - 3.5x^2$$
$$0 = 2.5x - 3.5x^2$$
$$0 = x(2.5 - 3.5x)$$
$$x = 0 \_ or \_ x = \frac{2.5}{3.5} = \frac{5}{7} \approx 0.714$$

Test your hypothesis by returning to the applet at
http://mathinsight.org/applet/function_iteration_cobweb_combined

12. Plug in $f(x) = 3.5 * x * (1-x)$ and a seed value of 0.04 for x_0. Describe the pattern that emerges as you iterate. Is it what you expected?

The time-series plot shows outputs bouncing between 4 values, rather than stabilizing at $\frac{5}{7}$ as we predicted. This is a surprise! The values in the cycle appear to be roughly 0.875, 0.825, 0.5, and 0.38.
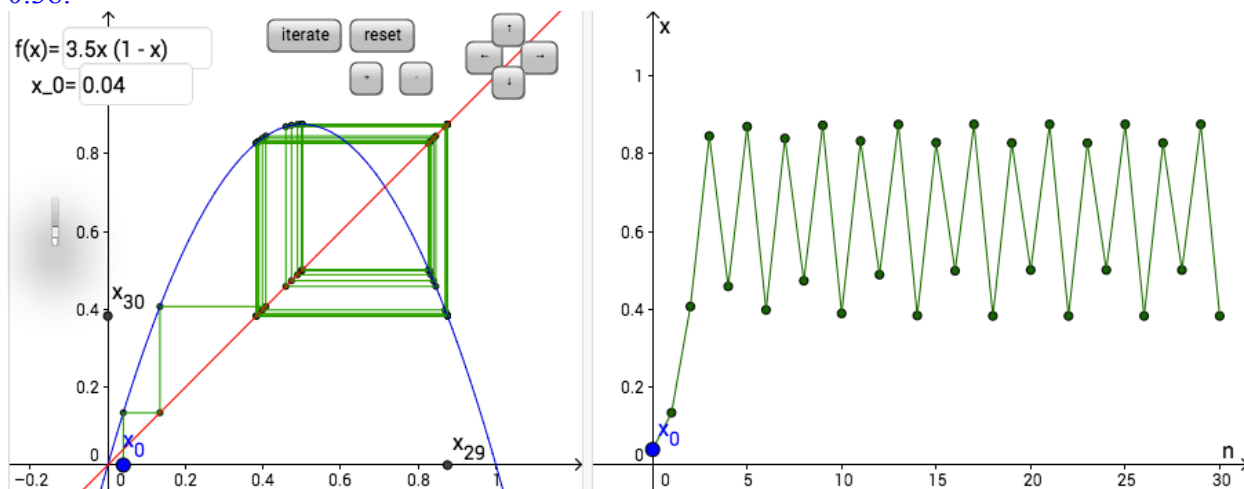
13. Mathematically, this system, as was the linear system before, is a **discrete dynamical system**. Discrete means that it moves in chunks, from time 0 to time 1, rather than continuously along from 0 to 1. Look up "dynamical system" and put the definition into your own words below.
A dynamical system is one in which we can describe the current state of the system and a rule, often a function, that tells us, given the current state, how to compute what the next state will be.

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As you can tell from just the 3 examples so far, the behavior of discrete dynamical systems is highly dependent on both initial value and the values of parameters in the equations. It is not possible, even for this relatively simple model, to create an explicit formula that will tell us exactly what proportion of the population will be sick at any given time; we can only march through each iteration one by one. Return to the applet and see if you can find an r value for which $x(t)$ cycles between 2 distinct values in the long term. Once you have one, find a second one. Hint: if you don't press "reset" when you change $f(x)$, the number of iterations from your previous function will remain. Also try to find an r value that produces a system in which no discernible pattern emerges. In our case, we'll see no discernible pattern for the proportion of the population with the disease. To find out more, see this link.

Many answers are possible to the questions below. Values that exhibit particularly interesting behavior can be found for r = 3.3, 3.6, 3.7, 3.8, 3.9, 4, and above 4.

14. The system oscillates between 2 values when $r =$ _____. The values are_____ and _____.

15. The system oscillates between 3 or 4 values when $r =$ _____.

The values are_____and_____and_____and_____.

16. The system does not exhibit any discernible pattern when $r =$ _____ or_____

17. Compare the r values you've found with your classmates. Try to spot patterns of long-term behavior based on r value.

Students may spot some patterns, but it is unlikely that, with this small sample size, they will observe for themselves Feigenbaum's patterns on bifurcation. You should show this to them.

D. The Continuous Case

Interestingly, the seeming disorder inherent in this system disappears if we use *continuous* time rather than *discrete* time. In fact, in continuous time, this same initial set up has the **logistic growth** function as its explicit solution. Depending on the application, the logistic function will take on various forms. Your text uses the form $N(t) = \dfrac{c}{a + be^{-rt}}$ where $y = \dfrac{c}{a}$ is the horizontal asymptote of the graph, sometimes called the carrying capacity, and r represents the initial growth rate. We'll see more of this function later in the year. For now, to see how well a logistic curve fits the well-behaved data from functions with low r value, go back to your calculator document and find the page with the time-series graph for $r = 1.5$. Click the "Menu" button, then select "4: Analyze", then "6: Regression" and scroll down to select "B: Logistic Regression (d=0)."

18. Write down the logistic equation that fits your data for $r = 1.5$.

$y = \dfrac{0.3358477}{1 + 8.798538e^{-0.500562x}}$

19. Repeat this process and write down the logistic equation that fits your data for $r = 2.5$.

$$y = \frac{0.602698477}{1 + 32.7452e^{-1.503914x}}$$

Please note that the tools of modeling you've used here:
- recursive equations,
- tables,
- time-series plots,
- cobweb diagrams, and
- explicit equations when available

all help you understand the system you're modeling in slightly different ways, and that a progression through multiple tools is often the best way to really understand a system.

APPENDIX C: Unit 3 Handouts

Asymptotes

Let $f(x) = \dfrac{n(x)}{d(x)}$, where P is the degree of $n(x)$ and A is the leading coefficient of $n(x)$ and Q is the degree of $d(x)$ and B is the leading coefficient of $d(x)$.

I. HORIZONTAL ASYMPTOTES: If

    a. $P > Q$, then $f(x)$ has no horizontal asymptote.

    b. $P = Q$, then $y = \dfrac{A}{B}$ is the horizontal asymptote of $f(x)$.

    c. $P < Q$, then $y = 0$ (x-axis) is the horizontal asymptote of $f(x)$.

II. VERTICAL ASYMPTOTES: Find the vertical asymptotes by setting $d(x)$ equal to zero.

(**Note:** This assumes that $n(x)$ and $d(x)$ share NO zero….this would

complicate things, so we'll deal with it separately).

If $(x - c)^m$ is a factor of the denominator, then the graph of $f(x)$ has a vertical

asymptote at *x = c*. To determine behavior near the asymptote, we need the

multiplicity of the zero.

- If the multiplicity *m* is even, then the behavior at the vertical asymptote is the

  same on both sides of it (i.e. the graph goes up to positive infinity or down to

  negative infinity on both sides of the asymptote).

- If the multiplicity *m* is odd, then the behavior on the two sides of the

  asymptote

  is different (i.e. the graph goes to positive infinity on one side, and to negative

  infinity on the other side).

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III.   SLANT/OBLIQUE/FUNCTIONAL ASYMPTOTES: We can determine long-term behavior of rational functions by performing polynomial division.

- If P = Q+1, then the long-term behavior will be to approach a slant asymptote (linear).

- If P = Q+2, then the function's long-term behavior will be to approach a quadratic.

- This trend continues for larger disparities also.

IV.   REMOVABLE DISCONTINUITIES (aka HOLES): If $n(x)$ and $d(x)$ share a zero of the same multiplicity, the graph will have a removable discontinuity. The x-coordinate of the hole will be that of the shared zero. To find the y-coordinate of the discontinuity, cancel the shared factor and plug the x value into the reduced function.

MA451 Mini-unit on inequalities

You're all familiar with solving linear inequalities, but we turn now to solving polynomial and rational inequalities. In these cases, ***it is easiest to compare with 0*** (is the function positive or negative), rather than other numbers or terms. As such, the first step in solving is to move all of the terms to one side. Once you have done this, factor that non-zero side of the inequality. This can be more complicated for rational inequalities than for polynomials. Once you have done this, solve for the boundary points (where is the function equal to 0? Where is the denominator equal to 0?). These points divide the number line into intervals that are potential solution regions. Once you have divided the number line into possible solution intervals, you have several options:

- **Option 1** (which your book presents): **Make a table or diagram**
  - o In this method, you choose an *x*-value that lies in each interval and plug it in to determine whether each individual factor is positive or negative. Then check whether the result of multiplying/dividing these positive and negative terms is positive or negative.
- **Option 2**: **Use what you know from polynomial and rational graphs**
  - o In this method, as with graphing polynomial and rational functions, you need only 1 particular value that does not lie on the *x*-axis. It may be the y-intercept or it may be another point that you need to find. Once you have this point, use what you know about multiplicities of zeros and asymptotes to fill in behavior in the remaining intervals. The book does not present this alternative, because this textbook section comes in Chapter 1, but you may find it easier based on the work you've already done in Chapter 3.

Textbook support can be found by reading pp. 74-77. Problems for this section are on p. 80 (#35-72, 99-103, 113-119). Your teacher will assign a subset of these problems to you.

APPENDIX D: Unit 4 Handouts, Activities, and Solutions
MA 451 Properties of Exponents and Logarithms

Rules of Exponents

Rules of Logarithms

$x^m x^n = x^{m+n}$

$\dfrac{x^m}{x^n} = x^{m-n}$

$y = \log_b(x) \Leftrightarrow b^y = x$

$\left(x^m\right)^n = x^{mn}$

$\log_b\left(b^x\right) = x$

$x^{-n} = \dfrac{1}{x^n}$

$b^{\log_b(x)} = x$

$(xy)^n = x^n y^n$

$\log_b(1) = 0$

$\left(\dfrac{x}{y}\right)^n = \dfrac{x^n}{y^n}$

$\log_b(b) = 1$

$\log(x) = \log_{10}(x)$

$x^{1/n} = \sqrt[n]{x}$

$\ln(x) = \log_e(x)$

$x^{m/n} = \sqrt[n]{x^m} = \left(\sqrt[n]{x}\right)^m$

$\log_b(xy) = \log_b(x) + \log_b(y)$

$\sqrt[n]{xy} = \sqrt[n]{x}\sqrt[n]{y}$

$\log_b\left(\dfrac{x}{y}\right) = \log_b(x) - \log_b(y)$

$\sqrt[n]{\dfrac{x}{y}} = \dfrac{\sqrt[n]{x}}{\sqrt[n]{y}}$

$\log_b\left(x^n\right) = n \cdot \log_b(x)$

$\sqrt[m]{\sqrt[n]{x}} = \sqrt[n]{\sqrt[m]{x}} = \sqrt[mn]{x} = x^{1/(mn)}$

$\log_b(x) = \dfrac{\log_a(x)}{\log_a(b)}$

Things to remember in general:
Logarithmic functions and exponential functions are inverses of one another. To "undo" an exponential function and bring the variable out of the exponent, you need to take the logarithm of both sides. It doesn't matter what base you choose, but it is easiest if the base of the logarithm matches the base of the exponential function.
If you'd like to "undo" a logarithm, you should rewrite the logarithm in exponential form (see the first rule of logarithms, above).
The change of base formula (last rule of logarithms) is becoming obsolete as technology (your TI-Nspire, among others) makes it possible to easily take a logarithm with any base. When solving more complicated equations, it helps to get both sides into a single exponential function (or a single logarithmic function, depending on what was asked) before trying to undo the exponential (logarithmic) function.
It also helps to remember that if $b \neq -1$, 0, or 1, then $b^m = b^n \Leftrightarrow m = n$, and that, if $x \neq 0, a > 0, b > 0$, then $a^x = b^x \Leftrightarrow a = b$.

NAME:
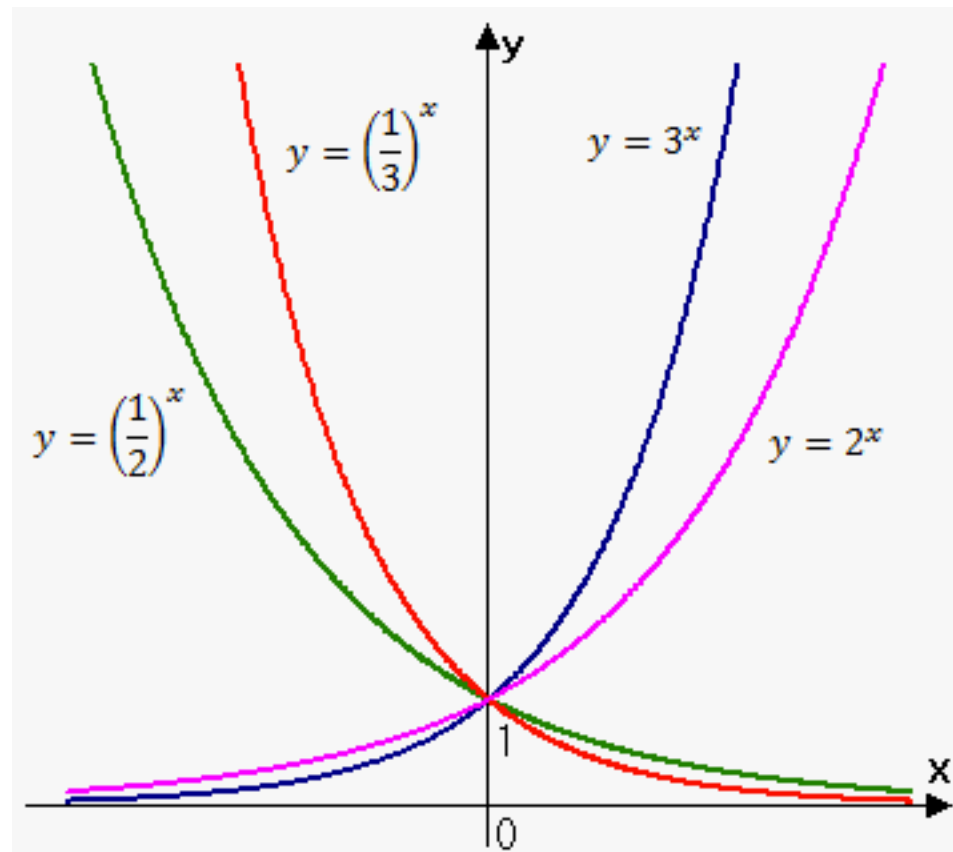
What do you notice about these graphs of exponential functions?

Graph $y = \dfrac{1}{2} \cdot 3^x$ without your calculator.

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

Where,

- P = principal amount (initial investment)
- r = annual nominal interest rate (as a decimal)
- n = number of times the interest is compounded per year
- t = number of years

Do you remember how this works?

If you invest \$2,500 at $6\frac{3}{4}\%$ compounded monthly, how much will you have after seven years?

How long will it take for you to double your money at $8\frac{1}{4}\%$ compounded weekly?

Find the annual percentage yield for an investment that earns $5\frac{1}{2}\%$ per year, compounded quarterly.

Here's our continuous compounding formula:

$$A = P e^{rt}$$

← time ← interest rate

final amount → A

principle (initial amount) → P

Let's do an example:

If you invest **$1,000,000** in an account paying **12%** compounded continuously, how much will you have in the account after **20** years?

$$A = P e^{rt}$$

$$A = 1000000\, e^{.12\,(20)} \approx \$11,023,176.38$$

But wait, what was that *e* thing?

In your own words describe *e*:

It is not easy to do is it?

Graph $y = e^x$ and $y = 2^x$ on the same axis.

The number e is defined as the value that $\left(1+\dfrac{1}{x}\right)^x$ approaches as x becomes large. (In Calculus, we will use the concept of limit to define e as $\displaystyle\lim_{x\to\infty}\left(1+\dfrac{1}{x}\right)^x$) Here is the table of values:

x	$\left(1+\dfrac{1}{x}\right)^x$
1	2.00000
5	2.48832
10	2.59374
100	2.70481
1,000	2.71692
10,000	2.71815
100,000	2.71827
1,000,000	2.71828

So, what is so special about e?

Well, it occurs in our world more than you would think.

It shows up when we compound interest continually – like the example we just did. Base-e exponential functions have an advantage when you study calculus because the rate of change of e^x is e^x. The natural number e is even in the Gateway Arch in St. Louis, MI - the arch can be defined as $y = a(e^{bx} + e^{-bx})$.

What do you notice about these graphs of exponential functions?
All are concave up. All go through (0,1). If the base >1, they're monotone increasing and have a
horizontal asymptote at y=0 as x heads to negative infinity. If 0<base<1, they're monotone
decreasing and have a horizontal asymptote at y=0 as x heads to positive infinity. For base>1, a
larger base means a steeper rate of increase for positive x values. For 0<base<1, a base closer to
0 means a steeper rate of decrease for negative x values.

Graph $y = \dfrac{1}{2} \cdot 3^x$ without your calculator.

Key points are $\left(0, \dfrac{1}{2}\right)$ and $\left(1, \dfrac{3}{2}\right)$

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$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

Where,

- P = principal amount (initial investment)
- r = annual nominal interest rate (as a decimal)
- n = number of times the interest is compounded per year
- t = number of years

Do you remember how this works?
Mostly, and I'd love to find these interest rate now!

If you invest \$2,500 at $6\frac{3}{4}\%$ compounded monthly, how much will you have after seven years?

$$A(7) = 2500\left(1 + \frac{0.0675}{12}\right)^{12 \cdot 7} = \$4004.69$$

How long will it take for you to double your money at $8\frac{1}{4}\%$ compounded weekly?

$$2P = P\left(1 + \frac{0.0825}{52}\right)^{52t}$$
$$2 = 1.006875^{52t}$$
$$\log_{1.006875}(2) = 52t$$
$$t \approx 1.95\, years$$

Find the annual percentage yield for an investment that earns $5\frac{1}{2}\%$ per year, compounded quarterly.

$$\left(1 + \frac{0.055}{4}\right)^4 = 1.056145 \text{ , so 5.6145\% APY}$$

Here's our continuous compounding formula:

$$A = Pe^{rt}$$

final amount ← A

principle (initial amount) ← P

time → t

interest rate → r

Let's do an example:

If you invest $1,000,000 in an account paying 12% compounded continuously, how much will you have in the account after 20 years?
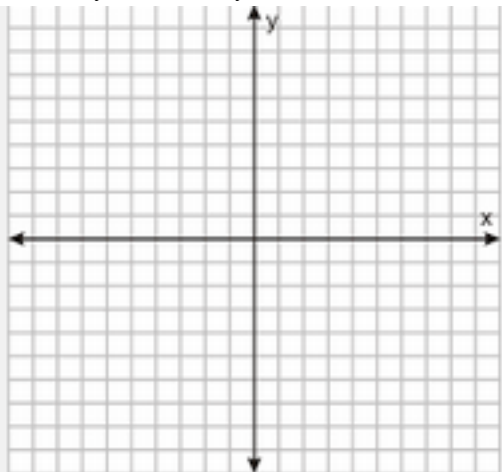
$$A = Pe^{rt}$$

$$A = 1000000\, e^{.12(20)} \approx \$11,023,176.38$$

But wait, what was that *e* thing? Just some letter that confuses me, because I can't just type it into my calculator, I need to hit a special button.

In your own words describe *e*: A transcendental number, formally the $\lim_{x \to \infty}\left(1 + \frac{1}{x}\right)^x$, whose value is about 2.718.

It is not easy to do is it? Not simply

Graph $y = e^x$ and $y = 2^x$ on the same axis.
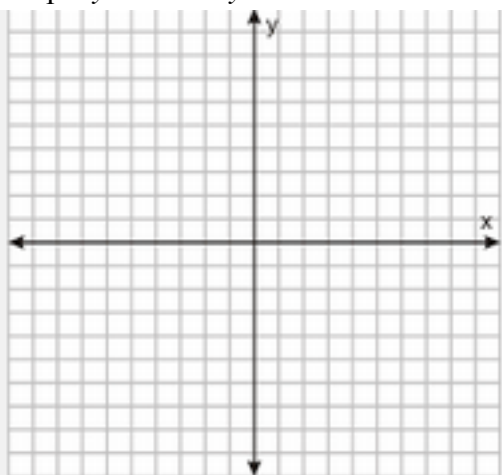
Both through (0,1), natural exponential steeper from there.

The number e is defined as the value that $\left(1+\dfrac{1}{x}\right)^x$ approaches as x becomes large. (In Calculus, we will use the concept of limit to define e as

$\displaystyle\lim_{x\to\infty}\left(1+\dfrac{1}{x}\right)^x$) Here is the table of values:

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So, what is so special about e?

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It shows up when we compound interest continually – like the example we just did. Base-e exponential functions have an advantage when you study calculus because the rate of change of e^x is e^x. The natural number e is even in the Gateway Arch in St. Louis, MI - the arch can be defined as $y = a(e^{bx} + e^{-bx})$.

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MA451   Name_____
More Review of Exponential and Logarithmic Functions

You may work with your partner on this set of exercises.

1. Write the steps you would take to solve the following equations for "hippos." Do not actually solve; just describe the necessary steps and why you would take them.
a) dogs + e^{hippos*cats} = horses

b) ln(hippos-gnus)+2ln(hippos+squirrels) = ln(elephants)

2. Rewrite the exponential equation in logarithmic form:
a) 4^3=64   b) $36^{1/2}$=6   c) 10^5=100000

3. Rewrite the logarithmic equation in exponential form:
a) $\log_2 32 = 5$ b) $\ln x = y$ c) $\log_3 27 = 3$

4. Expand the logarithm and write it as a sum/difference of two or more logarithms. Show your intermediate steps.
$\log(\frac{3x^2}{y^4})$

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5. Condense and write as one logarithm. Show your intermediate steps.
$\ln 5 + 3\ln z - 4\ln w$

6. Solve for x.

a) $\log_2(x+6) = 3$ 　　　　b) $e^x = 5$ 　　　　c) $e^{2x} + 5e^x - 24 = 0$

d) $\ln x + \ln(x+2) = \ln(x+6)$

7. Sketch (on a separate page) the graphs of y=e^x, y=e^{-x}, y=ln$x$. Be sure to label any and all intercepts (x or y), and any vertical or horizontal asymptotes. Include the domain and range of each function.

Name____Key_____
More Review of Exponential and Logarithmic Functions

You may work with your partner on this set of exercises.

1. Write the steps you would take to solve the following equations for "hippos." Do not actually solve; just describe the necessary steps and why you would take them.
a) dogs + e^{hippos*cats} = horses
Isolate the exponential part, which contains the variable.
Rewrite the exponential function as a logarithm [take the ln of both sides] to bring the variable out of the exponent.
Use algebra to isolate the desired variable.

b) ln(hippos-gnus)+2ln(hippos+squirrels) = ln(elephants)
Combine the left-hand side into a single logarithm.
Use the fact that ln(a) = ln(b) iff a = b to remove the logarithms.
Use algebra to isolate the desired variable.

2. Rewrite the exponential equation in logarithmic form:
a) 4^3=64   b) $36^{1/2}$=6   c) 10^5=100000

$\log_4(64) = 3$ $\log_{36}(6) = \dfrac{1}{2}$ $\log(100000) = 5$

3. Rewrite the logarithmic equation in exponential form:
a) $\log_2 32 = 5$ b) $\ln x = y$ c) $\log_3 27 = 3$

$2^5 = 32$ $e^y = x$ $3^3 = 27$

4. Expand the logarithm and write it as a sum/difference of two or more logarithms. Show your intermediate steps.
$\log(\dfrac{3x^2}{y^4})$
$= \log(3x^2) - \log(y^4) = \log(3) + \log(x^2) - 4\log(y) = \log(3) + 2\log(x) - 4\log(y)$

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5. Condense and write as one logarithm. Show your intermediate steps.

$\ln 5 + 3\ln z - 4\ln w$

$= \ln(5) + \ln(z^3) - \ln(w^4) = \ln\left(\dfrac{5z^3}{w^4}\right)$

6. Solve for x.

a) $\log_2(x+6) = 3$

$2^3 = x+6$
$8 = x+6$
$2 = x$

b) $e^x = 5$

$x = \ln(5)$

c) $e^{2x} + 5e^x - 24 = 0$

$Let\_u = e^x$
$u^2 + 5u - 24 = 0$
$(u+8)(u-3) = 0$
$u = -8\_or\_u = 3$
$e^x = -8\_or\_e^x = 3$
$\varnothing\_or\_x = \ln(3)$

d) $\ln x + \ln(x+2) = \ln(x+6)$

$\ln(x(x+2)) = \ln(x+6)$
$x^2 + 2x = x+6$
$x^2 + x - 6 = 0$ but the first solution is not in the domain, so the only solution is
$(x+3)(x-2) = 0$
$x = -3\_or\_x = 2$
x=2.

7. Sketch (on a separate page) the graphs of y=e^x, y=e^{-x}, y=lnx. Be sure to label any and all intercepts (x or y), and any vertical or horizontal asymptotes. Include the domain and range of each function.

The Fractal Coastline Problem

In the dynamical systems lab, you saw how following a simple set of rules through many iterations led to interesting and sometimes surprising results. This investigation will take you through a similar process of repetition of simple steps.

Background:

In the late 1940s, while studying the causes of conflict between countries, Lewis F. Richardson, a British mathematician, discovered that countries determined the measure of their common borders differently. In fact, he discovered that the measures were significantly different, depending on the length of the "ruler" used to estimate the coastline on a map. In particular, data he collected suggested that the length of a coastline becomes longer and longer as the ruler used gets smaller and smaller. He found this observation counter-intuitive, as for some object to have a well-defined length, the more precise ruler we use the closer we should get to a number that represents the length.

Benoit Mandelbrot, a French-American mathematician born in Poland, used Richardson's work as the basis of his study of the geometry of coastlines. His famous paper on the topic, *"How Long Is the Coast of Britain? Statistical Self-Similarity and Fractional Dimension"* was published in *Science* in 1967. The name *fractal* comes from the Latin word *fractus* which means broken or fractured.

Mandelbrot realized that it is meaningless to talk about the length of a coastline in the usual sense, as it is not well defined. Some other way of quantifying the coastline was needed, and this is what we will start exploring in this lab. We will only scratch the surface of fractal geometry, but we may revisit this topic later when we know more about iterative methods, transformations and matrices.

Instead of taking measurements on an actual map, we will use computer technology, which makes the process significantly easier. The Java applet used for this investigation, which creates an artificial coastline, can be found at the Boston University Polymer Studies Site:
http://polymer.bu.edu/java/java/coastline/coastlineapplet.html

Part A: Data gathering

1. Boot up the Fractal Coastline Applet using Safari or Firefox (not Chrome). This may take a couple of minutes. From the buttons in the middle of the page, choose *Automatic* and leave the *Roughness* setting at 0.2. Click on *Draw* at the top of the page.

    Press *Reset* and choose *Step by Step.* Repeatedly click on *Draw* and observe how the coastline is created. Describe your observations here.

2. The *roughness* is not a mathematical parameter, but one that controls how the algorithm constructs the coastline model. Change the roughness to 0.5 and create a new coastline. What do you think roughness controls? Describe in your own words.

3. Refresh the page to return to a roughness setting of 0.2 and automatic mode. Draw a new coastline. Compare your coastline to the others in the class. What do you observe?

4. We will now measure the coastline. Click on *Measure* (just below the roughness setting) and select the *Grid Measure* button. What we want to do is to count the number of squares in the grid that are needed to cover the coastline. While you are welcome to do this by hand (*Manual Measure*), it will be much more efficient to *Auto Measure*. When you click the *Auto Measure* button, the squares containing any piece of coastline will turn red, and the number of red squares will automatically be counted and stored in a table (not on this screen). Count the number of squares by hand just this once and write down the number here.

   Number of red squares = _____

   The size of the squares in the grid is given by *Length* at the top of the page. For the default length of 20, you counted a certain number of squares. To view the table where this ordered pair is stored, click on *Done Measuring*, and this will take you back to the coastline view. Here, click *Table*, and verify that the correct number of squares is displayed next to the 20.

5. Let's investigate how the number of boxes needed to cover the coastline changes with the size of the boxes. Click on *Coastline* to return to the picture of the coastline and click *Measure* again. Change the size of the squares in your grid by manipulating *Length* at the top of the page. Click on *Auto Measure* for a variety of lengths. Make some smaller than 20 and some larger than 20. Do at least 10 measurements.

The squares containing coastline will again turn red, and the count of squares will again be stored in the table. How do you expect changing the side length of the grid boxes will impact the count?

6. When you have at least 10 measurements, click *Done Measuring*. You will be returned again to the coastline view with the grid removed. Click on *Table* and copy down the ordered pairs here.

Grid size	Box count

Make some conjectures: What happens to the box count for large grid sizes? What is the trend as we shrink the size of the grid?

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7. Repeat the data collection phase for three different roughness settings. Try at least one roughness setting less than 0.2 and at least one greater than 0.2. Enter the ordered pairs in the tables below.
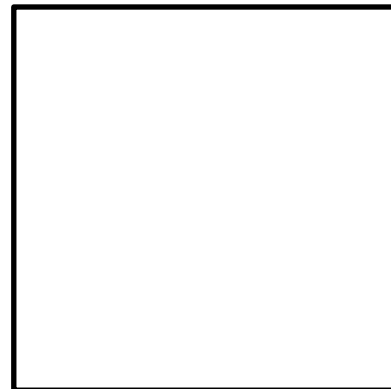
Roughness:

Grid size	Box count

Roughness:

Grid size	Box count

Roughness:

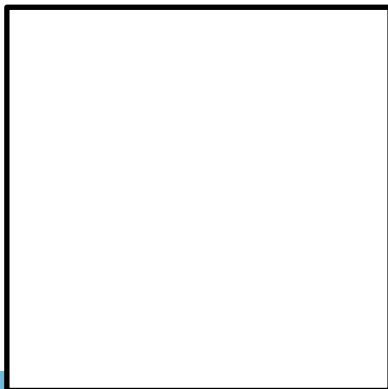| Grid size	Box count

8. The software can generate coastlines with a roughness between 0.05 and 0.9. What do you think: what would a coastline with a roughness of zero look like? How about a roughness of 1? Draw such curves in the boxes and include a short description.

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The Fractal Coastline Problem

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Background:

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http://polymer.bu.edu/java/java/coastline/coastlineapplet.html

Getting the applet up and running may take some time: Java updates and permissions may need to be changed.

Part A: Data gathering

9. Boot up the Fractal Coastline Applet using Safari or Firefox (not Chrome). This may take a couple of minutes. From the buttons in the middle of the page, choose *Automatic* and leave the *Roughness* setting at 0.2. Click on *Draw* at the top of the page.

    Press *Reset* and choose *Step by Step.* Repeatedly click on *Draw* and observe how the coastline is created. Describe your observations here.
    At each step, each line segment is broken into 2, and the 2 pieces swing in or out. After n steps, there are 2^n line segments and $2^n + 1$ points.

10. The *roughness* is not a mathematical parameter, but one that controls how the algorithm constructs the coastline model. Change the roughness to 0.5 and create a new coastline. What do you think roughness controls? Describe in your own words.

Roughness controls how far the line segments swing at each iteration. It seems to govern the maximum allowed swing, as not all coastlines with the same roughness look identical. Smaller roughness settings yield smaller swings (smoother overall), while larger roughness settings yield larger swings and more convoluted coastlines.

11. Refresh the page to return to a roughness setting of 0.2 and automatic mode. Draw a new coastline. Compare your coastline to the others in the class. What do you observe?

All are different, but they are similar in how smooth the coastline itself looks.

12. We will now measure the coastline. Click on *Measure* (just below the roughness setting) and select the *Grid Measure* button. What we want to do is to count the number of squares in the grid that are needed to cover the coastline. While you are welcome to do this by hand (*Manual Measure*), it will be much more efficient to *Auto Measure*. When you click the *Auto Measure* button, the squares containing any piece of coastline will turn red, and the number of red squares will automatically be counted and stored in a table (not on this screen). Count the number of squares by hand just this once and write down the number here.

Number of red squares = _This will vary. Most likely it will be in the 50-60 range.__

The size of the squares in the grid is given by *Length* at the top of the page. For the default length of 20, you counted a certain number of squares. To view the table where this ordered pair is stored, click on *Done Measuring*, and this will take you back to the coastline view. Here, click *Table*, and verify that the correct number of squares is displayed next to the 20.

13. Let's investigate how the number of boxes needed to cover the coastline changes with the size of the boxes. Click on *Coastline* to return to the picture of the coastline and click *Measure* again. Change the size of the squares in your grid by manipulating *Length* at the top of the page. Click on *Auto Measure* for a variety of lengths. Make some smaller than 20 and some larger than 20. Do at least 10 measurements.

The squares containing coastline will again turn red, and the count of squares will again be stored in the table. How do you expect changing the side length of the grid boxes will impact the count?
I'll need more boxes when they're smaller, and fewer when they're bigger.

14. When you have at least 10 measurements, click *Done Measuring*. You will be returned again to the coastline view with the grid removed. Click on *Table* and copy down the ordered pairs here.

Grid size	Box count
5	384
8	205
10	155
12	114
14	103
17	83
20	65
21	61
24	48
29	36
32	38
38	30
50	17
60	15
65	16
72	12
78	12
85	9
95	9

Make some conjectures: What happens to the box count for large grid sizes? What is the trend as we shrink the size of the grid?
Larger grid means fewer boxes, smaller grid means more boxes, but it's not a clear trend of

inverse proportionality. $(gridsize)(boxcount) \neq c$
$(gridsize)^2 (boxcount) \neq c$

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15. Repeat the data collection phase for three different roughness settings. Try at least one roughness setting less than 0.2 and at least one greater than 0.2. Enter the ordered pairs in the tables below. Students may not be able to see the Roughness until they measure and come back to the map. Record-keeping is important; one cannot copy and paste from the applet.

Roughness: 0.4

Grid size	Box count
5	780
7	500
15	174
21	100
22	97
37	38
42	33
56	23
66	20
78	15
87	12
10	295
13	204
29	59

Roughness: 0.65

Grid size	Box count
5	1292
7	803
10	438
13	292
15	231
20	140
26	97
31	67
38	53
49	37
56	35
62	27
73	20
85	15

Roughness: 0.12

Grid size	Box count
5	271
8	152
13	87
19	56
22	45
28	36
38	24
45	19
56	17
66	15
72	14
78	12
88	9
95	10

16. The software can generate coastlines with a roughness between 0.05 and 0.9. What do you think: what would a coastline with a roughness of zero look like? How about a roughness of 1? Draw such curves in the boxes and include a short description.

Roughness 0 would have no swing and be a straight line segment drawn vertically somewhere other than the middle of the box.

Roughness 1 would be so crazy a zig-zag that, no matter how small a grid we take, there's always a piece of coastline in every box.

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Part B: Graphing the data

Although the Fractal Coastlines applet can graph the data you just collected and do a curve fit, it is important to understand why it fits that specific curve to the data. We also want to become more comfortable with the TI-Nspire, the tool you'll be using most heavily in this class. With that in mind, we'll use the TI-Nspire to explore the problem.

If you need help with the TI-NSpire, online video tutorials can be found at
http://www.atomiclearning.com/ti_nspire .

17. Open a new Document. Your first page should be a Lists and Spreadsheets page. Put values of the independent variable (grid size, which you controlled) in the first column and name it *grid*. Put values of the dependent variable (the box count which depended on the chosen grid size) in the second column and name it *boxcount*. You should use your data from roughness 0.2 first.

18. Add a Data and Statistics page. Plot the data. Visually inspect the data and describe the shape of a curve that you believe would fit the points well. You should use concepts like "increasing/decreasing", "concave up/down", and "asymptotic." Write this down.

19. Save your document as "yourinitials coastline #", replacing the # with the roughness level used to generate that data (ie. "nbw coastline 0_2" since the software does not like using a decimal point in the file name).

20. You collected data for 3 additional roughness settings during Part A. Create a new document for each roughness setting and repeat steps 1 to 3 for each. Please don't forget to save your documents with their own names.

21. Plot the data and visually inspect it. Has the shape of the curve that you believe would fit the data well changed with the various roughness settings? What are the similarities, what are the differences?

22. The smallest roughness you can set with the applet is 0.05. What would a coastline with roughness 0 look like? (see Part A for a hint) The size of the map in the applet is 400 by 400. Using various values for the box length, compute the number of boxes needed to cover the straight line segment. For simplicity, use box lengths that divide 400 evenly. Make a table just like in part A, then sketch the scatter plot.

23. Imagine we have an extremely rough coastline that seems to fill the map with zigzags. No matter how small the boxes we use are, there will always be a piece of the coastline in each box. The size of the map is 400 by 400. What is the number of boxes needed to cover this 400 by 400 square for various box lengths? For simplicity use box lengths that divide 400 evenly. Make a table, then sketch the scatter plot.

Part B: Graphing the data

Although the Fractal Coastlines applet can graph the data you just collected and do a curve fit, it is important to understand why it fits that specific curve to the data. We also want to become more comfortable with the TI-Nspire, the tool you'll be using most heavily in this class. With that in mind, we'll use the TI-Nspire to explore the problem.

If you need help with the TI-NSpire, online video tutorials can be found at http://www.atomiclearning.com/ti_nspire .

24. Open a new Document. Your first page should be a Lists and Spreadsheets page. Put values of the independent variable (grid size, which you controlled) in the first column and name it *grid*. Put values of the dependent variable (the box count which depended on the chosen grid size) in the second column and name it *boxcount*. You should use your data from roughness 0.2 first.

25. Add a Data and Statistics page. Plot the data. Visually inspect the data and describe the shape of a curve that you believe would fit the points well. You should use concepts like "increasing/decreasing", "concave up/down", and "asymptotic." Write this down.
The curve is decreasing and concave up. It appears to be asymptotic to both axes (x=0 and y=0). Perhaps a curve like $y = \dfrac{1}{x}$ would fit, though we found in Part A that it's not an inversely proportional relationship. Perhaps an exponential curve, since that's what we've been doing for homework, though those have no vertical asymptotes.

26. Save your document as "yourinitials coastline #", replacing the # with the roughness level used to generate that data (ie. "nbw coastline 0_2" since the software does not like using a decimal point in the file name).

27. You collected data for 3 additional roughness settings during Part A. Create a new document for each roughness setting and repeat steps 1 to 3 for each. Please don't forget to save your documents with their own names.

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28. Plot the data and visually inspect it. Has the shape of the curve that you believe would fit the data well changed with the various roughness settings? What are the similarities, what are the differences?

The general shape (decreasing, concave up, asymptotic to the axes) is similar but the steepness has changed. A different *y*-scale is needed to plot these, since a larger roughness yields a larger boxcount. The change in y-scale can be hard for students to see and notice; it is worth pointing out.

29. The smallest roughness you can set with the applet is 0.05. What would a coastline with roughness 0 look like? (see Part A for a hint) The size of the map in the applet is 400 by 400. Using various values for the box length, compute the number of boxes needed to cover the straight line segment. For simplicity, use box lengths that divide 400 evenly. Make a table just like in part A, then sketch the scatter plot. This is hard for many students and may need to be teacher-led. It helps to have them think about values near the extremes (1 box of sidelength 400, or boxes of sidelength 1) and work towards the middle with lengths that divide 400 nicely.

A coastline with roughness 0 is a straight line (no bend when split into 2 parts). For a 400x400 viewing window, 100X100 grid means 4 boxes, 50x50 means 8 boxes, 25x25 means 16 boxes, 20x20 means 20 boxes, 8x8 means 50 boxes, etc. Have students come up with 10 grid lengths that divide 400 nicely and compute the number of boxes needed. The scatter plot should follow $f(x) = \dfrac{400}{x}$, since the grids divide one length of the map exactly.

30. Imagine we have an extremely rough coastline that seems to fill the map with zigzags. No matter how small the boxes we use are, there will always be a piece of the coastline in each box. The size of the map is 400 by 400. What is the number of boxes needed to cover this 400 by 400 square for various box lengths? For simplicity use box lengths that divide 400 evenly. Make a table, then sketch the scatter plot.

Again, students may need help here. Guide them to think about the extreme cases and to use lengths that divide 400 evenly.

For this example, the function should follow $f(x) = \dfrac{400^2}{x^2}$, since every grid square has a piece of the coastline in it.
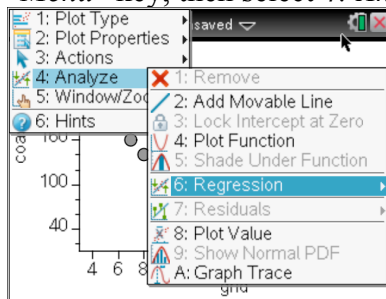
This gets students thinking about power functions, and about what curves might fit their data from roughness settings between 0 and 1. You might want to highlight that roughness 0 gives a 1-dimensional graph (straight line segment), which roughness 1 required a full 2-dimensional graph. These questions will be addressed in Part C of the activity.

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Part C: Fitting curves to the data

It is relatively easy to look at data and determine whether or not there is a linear pattern. It is much more challenging to look at non-linear data, even if it has only one type of behavior (decreasing and concave up, for example), and determine "how curved is it?" or "what type of function fits this data best?" We'll now examine ways to answer this question using the rules of exponential and logarithmic functions.

1.   On your calculator, open your document "coastline 0_2" and look at the scatter plot you created earlier. Clearly, the data is not linear. What types of functions might be a better fit? Given the decreasing, concave-up nature of the curve, we speculated that either an exponential model or a power model might be appropriate. Let's run both regressions, see how they look visually, and then examine them mathematically.

2.   On your original *Data and Statistics* page (the one with the scatter plot of data), click the "*Menu*" key, then select *4: Analyze*, then *6: Regression*, as shown in the screen shot below.
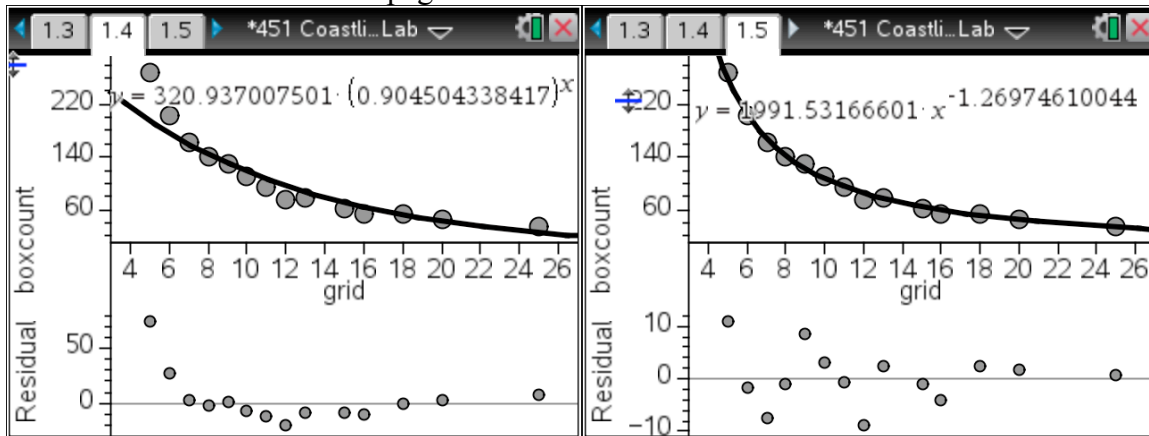

From the *6: Regression* menu, select *8: Exponential* and watch the calculator fit an exponential function to your data. Write down the function here:

3.   One tool that we will learn to use to help us decide whether a function is a good fit for data is a **residual plot**. Click *Menu-Analyze* again, and you will see *7: Residuals*. Select *2: Show Residual Plot*. Describe what you see.

4.   Because we want to compare exponential and power functions, we'll leave this page and create a new *Data and Statistics* page within the same document. Recreate the scatter plot of your data, then ask your calculator to "*7: Show Power*" regression. Once you have the regression, you can plot the residuals as you did above. Write down the equation of the power function here:

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Your two *Data and Statistics* pages should now look like this:

We need to try to choose a function that we think fits our data the best. Selecting an appropriate function allows us to **make predictions** about behavior for points other than our data.

In order to use Statistical inference, the residuals of a least-squares best-fit **line** to a data set should reflect only *random* errors in the model. The residual plot should look <u>noisy</u> (without apparent pattern), and the residuals should have equal spread around the best-fit line. A smaller scale on the residual plot is another potential indicator of a better fit.

Nonlinear data needs to be transformed before applying regression analysis. The rules of logarithms will be key to helping us transform our data so that we can use regression analysis. Your calculator performs this process every time you ask it to perform a non-linear regression.

5.　　Starting with an **exponential model** $y = a \cdot b^x$, we can take the logarithm (with any base) of both sides. Let's agree to use the natural log (aka: ln; log base *e*) for now. Use the rules to see if you can get to this statement: $\ln(y) = x\ln(b) + \ln(a)$.

6. Setting $Y=\ln(y)$, $B=\ln(b)$, and $A=\ln(a)$, we get $Y = Bx + A$, showing that $Y=\ln(y)$ becomes a linear function of x, with a slope of $\ln(b)$ and y-intercept of $\ln(a)$. A graph of $\ln(y)$ vs. x should be linear. On your NSpire spreadsheet, make a new column, the ln*(boxcount)* as shown below.

7. Add a new Data and Statistics page to your document. Plot ln(*boxcount*) on the y-axis vs. grid on the x-axis. Does the data appear to be linear? Run a linear regression to see how well a line fits. If a line is a good fit for this transformed data, then the original data can be modeled well by an exponential function. If the line is not a good fit, then the exponential model is not appropriate for our original data, and we should try a different model. This type of plot, with the logarithm of a variable on one axis and the other axis unchanged is called a *semi-log plot*. How well does the regression line fit the ln(*count*) vs. *grid* plot? Run the linear regression in your spreadsheet and write down the values of both the coefficient of determination (r^2) and the correlation coefficient (r). Would you be comfortable with this as your model?

8. To similarly test the validity of a **power function**, $y = a \cdot x^b$, as a model for our data, we again take the natural logarithm of both sides. Can you get to the statement $\ln(y) = b\ln(x) + \ln(a)$ using algebra?

9. In this case, ln (y) is a linear function in the variable ln (x). This is easiest to see if we let $A = \ln(a)$, $X = \ln(x)$, and $Y = \ln(y)$. The equation is then $Y = b \cdot X + A$. If we plot ln y vs. ln x, what should the slope of the line be? What is its y-intercept?

10.   In your NSpire spreadsheet, make a new column, ln(*grid*). Add a new *Data and Statistics* page to your document and plot ln(*count*) vs ln(*grid*). Does the data appear to be linear? Run a linear regression to see how well a line fits. <u>If a line is a good fit for this transformed data, then the original data can be modeled well by a power function. If the line is not a good fit, then the power model is not appropriate for our original data and we should try a different model.</u> This type of plot, with the logarithm of a variable on one axis and the logarithm of the other variable on the other axis is called a *log-log plot*. How well does the regression line fit the ln(*count*) vs. ln(*grid*) plot? Run the linear regression in your spreadsheet and write down the values of both the coefficient of determination (r^2) and the correlation coefficient (r). Would you be comfortable with this as your model?

Here is my plot of ln(*count*) vs. ln(*grid*).
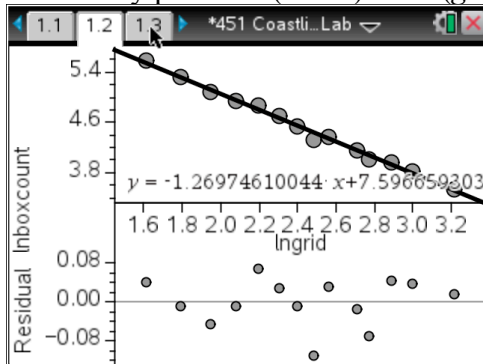
11. The line appears to fit the data well (residuals are small and scattered, $r^2 = 0.99338$, $r = -0.99668$), and I will choose this as my model. We should now transform this linear regression back into a power function for our original data. We need the rules of exponents to do this, starting in particular with the fact that if $a = b$, then $c^a = c^b$. We first recall that the linear equation is ln y vs. ln x for our original y and x, giving us the following linear function to transform:
ln y = -1.2697 (ln x) + 7.59665

$y = 1991.53\, x^{-1.2697}$

On day 2, when we first ran a power regression, this is exactly what the calculator gave us. Amazing! This is how the calculator does power and exponential regression. The idea is to linearize the data first and then convert it back.

If you were to have done a curve fit in the coastline applet, you would have gotten this result: $N = 1991.53L^{-1.26}$, where N is the box count of the coastline and L is the size of the grid squares.

12.   Repeat this process for another two of your roughness settings, transforming the linear function for your ln(*count*) vs. ln(*grid*) plot into a power function. Confirm that it matches (give or take rounding in your conversion) the initial power regression performed by the calculator. Fill in the remainder of the table with information from classmates who chose different roughness settings than you did.

Roughness	Linear equation	Calculated Power Equation	Power Regression

13. What patterns do you see? How does the roughness setting relate to the equations?

A straight line is 1-dimensional. A filled plane is 2-dimensionsal. All of the coastlines you created fall somewhere in between these 2 extremes. The roughness is a good proxy for the **fractal dimension**, a measure of how much a curve fills the plane. This idea can be expanded beyond 2 dimensions. For example, think about a sponge as filling almost, but not all, of the 3-dimensions.
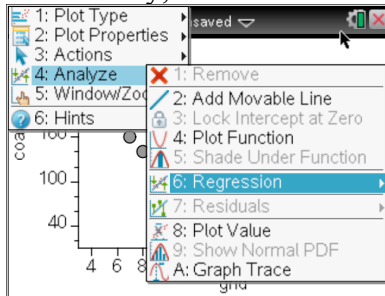
14. You utilize fractals every day; most cellphones now use fractal antennae. Perform a quick google search to see some typical patterns and read more. Write a few sentences at the top of the next page about what you find.

15. You should also search for an image of a Sierpinski carpet to see a pattern first described before World War I. Describe below the simple rules followed over and over to create this fractal.

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Part C: Fitting curves to the data
It is relatively easy to look at data and determine whether or not there is a linear pattern. It is much more challenging to look at non-linear data, even if it has only one type of behavior (decreasing and concave up, for example), and determine "how curved is it?" or "what type of function fits this data best?" We'll now examine ways to answer this question using the rules of exponential and logarithmic functions.

1.   On your calculator, open your document "coastline 0_2" and look at the scatter plot you created earlier. Clearly, the data is not linear. What types of functions might be a better fit? Given the decreasing, concave-up nature of the curve, we speculated that either an exponential model or a power model might be appropriate. Let's run both regressions, see how they look visually, and then examine them mathematically.

2.   On your original *Data and Statistics* page (the one with the scatter plot of data), click the "*Menu*" key, then select *4: Analyze*, then *6: Regression*, as shown in the screen shot below.

From the *6: Regression* menu, select *8: Exponential* and watch the calculator fit an exponential function to your data. Write down the function here:
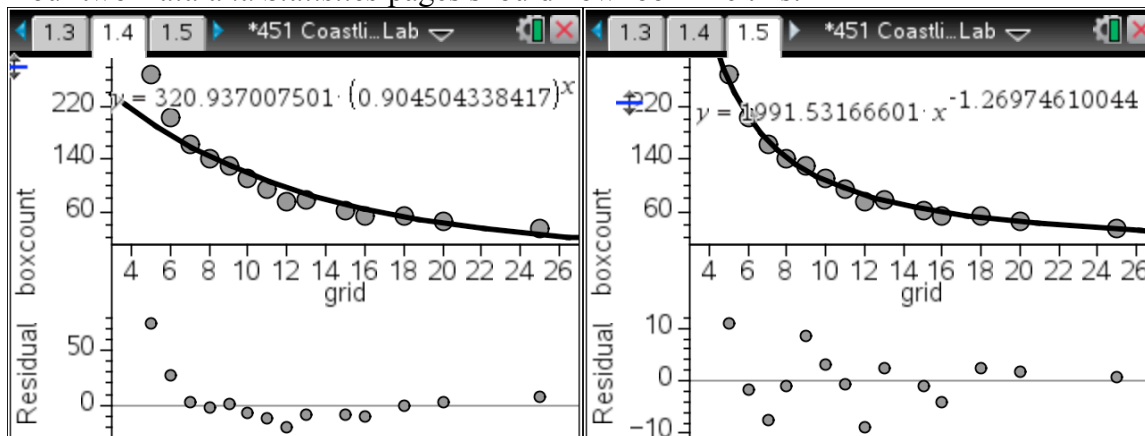
$$y = 168.044(0.964199)^x$$

3.   One tool that we will learn to use to help us decide whether a function is a good fit for data is a **residual plot**. Click *Menu-Analyze* again, and you will see *7: Residuals*. Select *2: Show Residual Plot*. Describe what you see.
The pattern of dots looks similar to the original (concave up, mostly decreasing, HA @ y=0), though it does dip below the *x*-axis for a while. The scale of the residual plot goes up to 250. The exponential decay function (regression line) does not come up steeply enough as *x* approaches 0.

4.   Because we want to compare exponential and power functions, we'll leave this page and create a new *Data and Statistics* page within the same document. Recreate the scatter plot of your data, then ask your calculator to "*7: Show Power*" regression. Once you have the regression, you can plot the residuals as you did above. Write down the equation of the power function here:
$y = 2966.51x^{-1.28205}$ This looks like a better fit; the behavior near the *y*-axis matches the data better; the scale of the residual plot is much smaller; there is no pattern evident in the residual plot.

Your two *Data and Statistics* pages should now look like this:

We need to try to choose a function that we think fits our data the best. Selecting an appropriate function allows us to **make predictions** about behavior for points other than our data.

In order to use Statistical inference, the residuals of a least-squares best-fit **line** to a data set should reflect only *random* errors in the model. The residual plot should look <u>noisy</u> (without apparent pattern), and the residuals should have equal spread around the best-fit line. A smaller scale on the residual plot is another potential indicator of a better fit.

Nonlinear data needs to be transformed before applying regression analysis. The rules of logarithms will be key to helping us transform our data so that we can use regression analysis. Your calculator performs this process every time you ask it to perform a non-linear regression.

5.   Starting with an **exponential model** $y = a \cdot b^x$, we can take the logarithm (with any base) of both sides. Let's agree to use the natural log (aka: ln; log base *e*) for now. Use the rules to see if you can get to this statement: $\ln(y) = x\ln(b) + \ln(a)$.

$$y = ab^x$$
$$\ln(y) = \ln(ab^x)$$
$$\ln(y) = \ln(a) + \ln(b^x)$$
$$\ln(y) = \ln(a) + x\ln(b)$$

It may be easier for students to see this as linear if you use capital letters to substitute for logarithms. That is, let $Y = \ln(y), A = \ln(a), B = \ln(b)$ so that $Y = A + Bx$, which is clearly linear.

6. Setting $Y=\ln(y)$, $B=\ln(b)$, and $A=\ln(a)$, we get $Y = Bx + A$, showing that $Y=\ln(y)$ becomes a linear function of x, with a slope of $\ln(b)$ and y-intercept of $\ln(a)$. A graph of $\ln(y)$ vs. x should be linear. On your NSpire spreadsheet, make a new column, the ln*(boxcount)* as shown below.

7. Add a new Data and Statistics page to your document. Plot ln(*boxcount*) on the y-axis vs. grid on the x-axis. Does the data appear to be linear? Run a linear regression to see how well a line fits. <u>If a line is a good fit for this transformed data, then the original data can be modeled well by an exponential function. If the line is not a good fit, then the exponential model is not appropriate for our original data, and we should try a different model.</u> This type of plot, with the logarithm of a variable on one axis and the other axis unchanged is called a *semi-log plot*. How well does the regression line fit the ln(*count*) vs. *grid* plot? Run the linear regression in your spreadsheet and write down the values of both the coefficient of determination (r^2) and the correlation coefficient (r). Would you be comfortable with this as your model?

$y = -0.036458x + 5.12423, r^2 = 0.881072, r = -0.9386$

No, I wouldn't be comfortable with this model. The scatter plot still seems curved, not linear. There is a clear pattern of data above the regression line on both ends and below the line in the middle.

8. To similarly test the validity of a **power function**, $y = a \cdot x^b$, as a model for our data, we again take the natural logarithm of both sides. Can you get to the statement $\ln(y) = b\ln(x) + \ln(a)$ using algebra?

$$y = ax^b$$
$$\ln(y) = \ln(ax^b)$$
$$\ln(y) = \ln(a) + \ln(x^b)$$
$$\ln(y) = \ln(a) + b\ln(x)$$

9. In this case, $\ln(y)$ is a linear function in the variable $\ln(x)$. This is easiest to see if we let $A = \ln(a)$, $X = \ln(x)$, and $Y = \ln(y)$. The equation is then $Y = b \cdot X + A$. If we plot $\ln y$ vs. $\ln x$, what should the slope of the line be? What is its y-intercept?

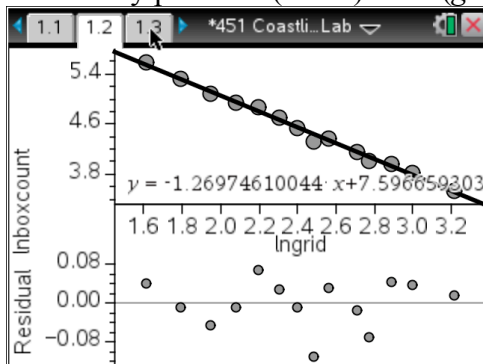The slope is b, the exponent of the original power function.

The y-intercept is A, the natural log of the original coefficient (vertical stretch factor).

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10. In your NSpire spreadsheet, make a new column, ln(*grid*). Add a new *Data and Statistics* page to your document and plot ln(*count*) vs ln(*grid*). Does the data appear to be linear? Run a linear regression to see how well a line fits. <u>If a line is a good fit for this transformed data, then the original data can be modeled well by a power function. If the line is not a good fit, then the power model is not appropriate for our original data and we should try a different model.</u> This type of plot, with the logarithm of a variable on one axis and the logarithm of the other variable on the other axis is called a *log-log plot*. How well does the regression line fit the ln(*count*) vs. ln(*grid*) plot? Run the linear regression in your spreadsheet and write down the values of both the coefficient of determination (r^2) and the correlation coefficient (r). Would you be comfortable with this as your model?

$y = -1.28205x + 7.99514, r^2 = 0.995928, r = -0.997962$

Yes, I'd be much more comfortable with this as my model.

Here is my plot of ln(*count*) vs. ln(*grid*).

11. The line appears to fit the data well (residuals are small and scattered, $r^2 = 0.99338$, $r = -0.99668$), and I will choose this as my model. We should now transform this linear regression back into a power function for our original data. We need the rules of exponents to do this, starting in particular with the fact that if $a = b$, then $c^a = c^b$. We first recall that the linear equation is ln y vs. ln x for our original y and x, giving us the following linear function to transform:

ln y = -1.2697 (ln x) + 7.59665

$e^{\ln y} = e^{-1.26971\ln x + 7.59665}$

$y = e^{-1.26971\ln x} e^{7.59665}$

$y = e^{\ln\left(x^{-1.26971}\right)}1991.51$

$y = 1991.51x^{-1.26971}$

$y = 1991.53\ x^{-1.2697}$

On day 2, when we first ran a power regression, this is exactly what the calculator gave us. Amazing! This is how the calculator does power and exponential regression. The idea is to linearize the data first and then convert it back.

If you were to have done a curve fit in the coastline applet, you would have gotten this result:

$N = 1991.53L^{-1.26}$, where N is the box count of the coastline and L is the size of the grid squares.

12.   Repeat this process for another two of your roughness settings, transforming the linear function for your ln(*count*) vs. ln(*grid*) plot into a power function. Confirm that it matches (give or take rounding in your conversion) the initial power regression performed by the calculator. Fill in the remainder of the table with information from classmates who chose different roughness settings than you did. Put 3 charts on the board for all groups to contribute. One chart should be for roughness = 0.2, one for values below 0.2, and one for roughness above 0.2. Encourage the inclusion of duplicate values.

Roughness	Linear equation	Calculated Power Equation	Exponent
0.2	$y = -1.28205x + 7.99514$	$y = 2966.51x^{-1.28205}$	-1.28205
0.4	$y = -1.46971x + 9.06573$	$y = 8653.59x^{-1.46971}$	-1.46971
0.65	$y = -1.55754x + 9.66374$	$y = 15736.5x^{-1.55754}$	-1.55754
0.9	$y = -1.67203x + 10.4077$	$y = 33113.6x^{-1.67203}$	-1.67203
0.12	$y = -1.12678x + 7.35327$	$y = 1561.29x^{-1.12678}$	-1.12678

13. What patterns do you see? How does the roughness setting relate to the equations?
As roughness increases, the vertical stretch increases and the magnitude of the power increases. There is more variation among the powers for large roughness settings than there is for smaller roughness settings. This isn't surprising, as roughness gave a maximum for amount of swing, and not all swings reached that maximum. There's more room for variation with a larger maximum swing.
The magnitude of the power is close to 1+roughness setting, though this relationship is not perfect.

A straight line is 1-dimensional. A filled plane is 2-dimensionsal. All of the coastlines you created fall somewhere in between these 2 extremes. The roughness is a good proxy for the **fractal dimension**, a measure of how much a curve fills the plane. This idea can be expanded beyond 2 dimensions. For example, think about a sponge as filling almost, but not all, of the 3-dimensions.

14. You utilize fractals every day; most cellphones now use fractal antennae. Perform a quick google search to see some typical patterns and read more. Write a few sentences at the top of the next page about what you find.

Fractals exhibit self-symmetry (parts look like each other and like the whole), and they can be created by iterating simple rules. Cell phones use fractal antennae because the self-similarity means they can accommodate multiple frequencies. They also take up less space than more traditional, directed antennae.

15. You should also search for an image of a Sierpinski carpet to see a pattern first described before World War I. Describe below the simple rules followed over and over to create this fractal.

Start with a colored square. Divide it into 9 congruent sub-squares (3x3 grid) and remove the central sub-square. Leave the removed area as white space.
Repeat this process with each of the 8 remaining sub-squares, splitting each into 9 congruent sub-sub-squares and removing the central one.
Repeat this process again with the remaining sub-sub-squares.
Continue.

APPENDIX E: Unit 5 and 6 Handouts, Activities, and Solutions
MA451 Probability Unit Motivation

You're a 43-year-old woman in good health. The day after your annual physical, you receive a call from your doctor. She tells you that your screening test for breast cancer came back positive. Because you're a reasonable person who remembers some of your high school math experience, you don't panic. Instead, you calmly ask your doctor: "what is the probability that I actually have breast cancer?"

Frighteningly enough, many doctors can't answer this question appropriately, at least those in a study published in Gerd Gigerenzer's *Calculated Risks*. In this study, Gigerenzer asked doctors a question similar to the one above and gave them this background information:

- The probability that a woman in this category has breast cancer is 1%.
- If a woman has breast cancer, the probability that she tests positive is 90%.
- If a woman does not have breast cancer, the probability that she tests positive anyway is 9%.

Girgerenzer gave the doctors 4 answers to choose from, and the doctors in the study answered less correctly among the answer choices than if they had been guessing at random.

In this unit, you'll learn several probability tools that will help you understand how to tell your doctor what the correct response is (10% for the situation above) and how she can calculate it for herself. Before we get to that point, we'll do a quick review of counting and probability tools and introduce some new ones.

In all of mathematics, clarity of wording is important; in probability it is critical. Likewise, it is always important to state your assumptions, or the ground rules within which you are working on a problem. The importance is magnified when dealing with probability questions.

Let's begin with a different scenario to make sure you're thinking clearly about how a question is asked. You have a pet fish in your dorm room, but you're leaving for the weeklong Thanksgiving vacation and not taking the fish with you. You ask a faculty child who lives in the dorm to take care of it for you. There's a 25% chance that the faculty child will forget to feed the fish. There is a 90% chance that the fish will die if it doesn't get fed during the week you're gone. There is a 15% chance that it will die during that week even if it does get fed.

Three questions come to mind, each with a slightly different focus:

1. What's the probability that your fish is alive when you come back to campus?
2. If the faculty child forgets to feed the fish, what is the probability that it's dead upon your return?
3. If the fish is dead when you return, what's the probability that the child forgot to feed it?

While the second and the third questions may sound very similar, they are decidedly different questions.

1. What is the "given information" in the second question?

2. What is the "given information" in the third question?

3. Which of these questions can you answer without doing any calculations? What is that answer?

Our work in this unit will focus on how to calculate and combine probabilities to answer the first question and the third one, and the answers to these questions require different skills. We'll also return to the breast cancer question and how you might answer it in your head pretty quickly. Along the way, you will refresh or learn the following skills/ideas:

- Counting principles
- Permutations and combinations
- Basic probability and the use of counting principles to compute probabilities
- The use of Venn diagrams, trees, and tables
- Independence of events
- Conditional Probability and natural frequencies as a way to estimate

In either this semester or next, you will also learn about expected value (mathematical expectation) and fair games. In January, you will learn about the binomial and geometric distributions (and maybe some others).

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- The probability that a woman in this category has breast cancer is 1%.
- If a woman has breast cancer, the probability that she tests positive is 90%.
- If a woman does not have breast cancer, the probability that she tests positive anyway is 9%.

Girgerenzer gave the doctors 4 answers to choose from, and the doctors in the study answered less correctly among the answer choices than if they had been guessing at random.

In this unit, you'll learn several probability tools that will help you understand how to tell your doctor what the correct response is (10% for the situation above) and how she can calculate it for herself. Before we get to that point, we'll do a quick review of counting and probability tools and introduce some new ones.

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Let's begin with a different scenario to make sure you're thinking clearly about how a question is asked. You have a pet fish in your dorm room, but you're leaving for the weeklong Thanksgiving vacation and not taking the fish with you. You ask a faculty child who lives in the dorm to take care of it for you. There's a 25% chance that the faculty child will forget to feed the fish. There is a 90% chance that the fish will die if it doesn't get fed during the week you're gone. There is a 15% chance that it will die during that week even if it does get fed.

Three questions come to mind, each with a slightly different focus:

4. What's the probability that your fish is alive when you come back to campus?
5. If the faculty child forgets to feed the fish, what is the probability that it's dead upon your return?
6. If the fish is dead when you return, what's the probability that the child forgot to feed it?

While the second and the third questions may sound very similar, they are decidedly different questions.

1. What is the "given information" in the second question?
The fac brat forgot to feed the fish.

2. What is the "given information" in the third question?
The fish is dead.

3. Which of these questions can you answer without doing any calculations? What is that answer?
The second one. 90% chance, which was given information in the opening paragraph.

Our work in this unit will focus on how to calculate and combine probabilities to answer the first question and the third one, and the answers to these questions require different skills. We'll also return to the breast cancer question and how you might answer it in your head pretty quickly. Along the way, you will refresh or learn the following skills/ideas:
- Counting principles
- Permutations and combinations
- Basic probability and the use of counting principles to compute probabilities
- The use of Venn diagrams, trees, and tables
- Independence of events
- Conditional Probability and natural frequencies as a way to estimate

In either this semester or next, you will also learn about expected value (mathematical expectation) and fair games. In January, you will learn about the binomial and geometric distributions (and maybe some others).

Counting Principles with "AND" and "OR"

Students at a fictional high school were polled about their favorite extreme sports. This table summarizes the information gathered:

	Freshman	Sophomore	Junior	Senior	Total
Hang gliding	50	35	33	29	
White water rafting	20	25	28	24	
Rock climbing	40	40	39	37	
Total					

1. Compute the column and row sums. How many students attend this high school?

2. Work with the numbers in the table. If a student is selected at random, what is the probability that

    a. … the student is a Junior? … the student is a Senior?

    b. … the student likes rock climbing?   … the student likes hang gliding?

    c. … the student is a Junior <u>and</u> likes rock climbing?

    d. … the student is a Junior <u>or</u> the student is a Senior?

    e. … the student is a Junior <u>or</u> the student likes rock climbing?

3. A student is selected at random. Define events such as
    J = "the randomly selected student is a Junior",
    RC = "the randomly selected student likes rock climbing",
    S = "the randomly selected student is a Senior",
    HG = "the randomly selected student likes hang gliding", etc.

    a. Are the events J and RC **mutually exclusive** or **overlapping**? Explain. Illustrate with a Venn diagram.

b. Are the events J and S **mutually exclusive** or **overlapping**? Explain. Illustrate with a Venn diagram.

c. Is it true that   P(J or S) = P(J) + P(S)? Why or why not? Explain your answer in everyday language.

d. Is it true that P(J or RC) = P(J) + P(RC)? Why or why not? Explain your answer in everyday language.

e. Complete the description of the **addition rule**:

If A and B are events then P(A or B) = _____.

In the special case when A and B are mutually exclusive, the rule becomes

P(A or B) = _____ because _____.

Here is the table again for your reference.

	Freshman	Sophomore	Junior	Senior	Total
Hang gliding	50	35	33	29	
White water rafting	20	25	28	24	
Rock climbing	40	40	39	37	
Total					

4.   Work with the numbers in the table. What is the probability that …

       a. a randomly selected student likes hang gliding?

       b. a randomly selected Senior likes hang gliding?

       c. a randomly selected student is a Senior?

       d. a randomly selected student who likes hang gliding is a senior?

       e. a randomly selected student is a Junior?

       f. a randomly selected student who likes rock climbing is a Junior?

5. In the question "what is the probability that a randomly selected Senior likes hang gliding" the sample space is reduced from all students in the school to the group of
_____.

This probability is called **conditional probability**, and we use the notation P(HG | S). Read: the probability of HG given S.

6. Compute the following conditional probabilities using the numbers in the table.

      a. P(HG | S) =   P(S | HG) =

      b. P(RC | J) =   P(J | RC) =

7.   How popular is hang gliding among Seniors, compared to the general school population?

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8.	How popular is rock climbing among Juniors, compared to the general school population? Contrast your answer with the answer to the previous question.

9.	We say that two events are **independent**, if the occurrence of one does not affect the probability of the other. If the occurrence of one does affect the probability of the other, the two events are **dependent**. Tell whether the following event pairs are dependent or independent. Explain why.

   a.	HG and S

   b.	RC and J

   c.	J and S

   Give the formal definition of independence using probabilities. Two events, A and B, are independent if _____ and _____ .

10.	How many students are Seniors <u>and</u> like hang gliding?

   What is the probability that a randomly selected student is a Senior <u>and</u> likes hang gliding? Express this probability as a ratio.

   We could think about selecting these students in two steps. First, select all the Seniors. Then, from the group of Seniors select the students who like Hang gliding.

   The probability that a randomly selected student is a Senior <u>and</u> likes hang gliding is

   $$P(S \text{ and } HG) = \frac{\# seniors}{\# total\_students} \bullet \frac{\# seniors\_who\_like\_hang\_gliding}{\# seniors} = \frac{}{400} \bullet \frac{29}{}$$

   Write the right-hand-side of the using probability notation.

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11.   What is the probability that a randomly selected student is a Junior and likes rock climbing? Express this probability as a ratio.

Think about selecting these students in two steps. First, select all the Juniors. Then, from the group of Juniors, select those who like rock climbing. Express this probability as a product of two ratios, then write it using probability notation.

What is the probability that a student likes rock climbing? Is this probability influenced by the student being a Junior (hint: what's the probability that a randomly selected Junior likes rock climbing)? Can P(J and RC) be expressed without conditional probabilities?

12. Complete the description of the **multiplication rule**:

If A and B are events then P(A and B) = _____

In the special case when the two events are independent, the rule becomes

P(A and B) = _____because_____.

13.   Write three examples (along with solutions) about the data in the table to share with your classmates. Be creative!

Counting Principles with "AND" and "OR"

Students at a fictional high school were polled about their favorite extreme sports. This table summarizes the information gathered:

	Freshman	Sophomore	Junior	Senior	TOTAL
Hang gliding	50	35	33	29	147
White water rafting	20	25	28	24	97
Rock climbing	40	40	39	37	156
TOTAL	110	100	100	90	400

1. Compute the column and row sums. How many students attend this high school?

400, assuming all responded

2. Work with the numbers in the table. If a student is selected at random, what is the probability that

   a. … the student is a Junior? … the student is a Senior?
   $$\frac{100}{400} = 0.25 \qquad\qquad \frac{90}{400} = 0.225$$

   b. … the student likes rock climbing?   … the student likes hang gliding?
   $$\frac{156}{400} = 0.39 \qquad\qquad \frac{147}{400} = 0.3675$$

   c. … the student is a Junior <u>and</u> likes rock climbing?
   $$\frac{39}{400} = 0.0975$$

   d. … the student is a Junior <u>or</u> the student is a Senior?
   $$\frac{100 + 90}{400} = 0.475$$

   e. … the student is a Junior <u>or</u> the student likes rock climbing?
   $$\frac{100 + 40 + 40 + 37}{400} = 0.5425$$

3. A student is selected at random. Define events such as
   J = "the randomly selected student is a Junior",
   RC = "the randomly selected student likes rock climbing",
   S = "the randomly selected student is a Senior",
   HG = "the randomly selected student likes hang gliding", etc.

   a. Are the events J and RC **mutually exclusive** or **overlapping**? Explain. Illustrate with a Venn diagram.

Overlapping, because there are 39 Juniors who like rock climbing. This is a good time to talk about filling in other values in a Venn diagram. It is also a good time to talk more about when to use the complement principle and when to simply compute. For example, what's left in the Junior category? Is it easier to add 33+28 to get 61 or to subtract 39 from 100?

b. Are the events J and S **mutually exclusive** or **overlapping**? Explain. Illustrate with a Venn diagram.

Mutually exclusive; a student cannot be both a Junior and a Senior.

c. Is it true that $P(J \text{ or } S) = P(J) + P(S)$? Why or why not? Explain your answer in everyday language.

Yes, this is true. Because there is no overlap between the categories, we can simply add the probabilities without "double-counting" anyone.

d. Is it true that $P(J \text{ or } RC) = P(J) + P(RC)$? Why or why not? Explain your answer in everyday language.

No, because there are 39 students who fall into both categories. If we simply add $P(J)+P(RC)$, we will count these students twice.

$P(J \text{ or } RC) = P(J) + P(RC) – P(J \text{ and } RC)$. This is formally known as the Inclusion-Exclusion Principle. Please make sure to give your students the formal name of the property. You may want to include generalization to the case of three overlapping categories. You should also include the notation of Union and Intersection of sets.

e. Complete the description of the **addition rule**:

If A and B are events then $P(A \text{ or } B) =$ ___$P(A) + P(B) – P(A \text{ and } B)$_____.

In the special case when A and B are <u>mutually exclusive</u>, the rule becomes

$P(A \text{ or } B) =$ ___$P(A) + P(B)$_____ because __$P(A \text{ and } B) = 0$_____.

Here is the table again for your reference.

	Freshman	Sophomore	Junior	Senior	TOTAL
Hang gliding	50	35	33	29	147
White water rafting	20	25	28	24	97
Rock climbing	40	40	39	37	156
TOTAL	110	100	100	90	400

4. Work with the numbers in the table. What is the probability that …

   a. a randomly selected student likes hang gliding?

   $$\frac{147}{400} = 0.3675$$

   b. a randomly selected Senior likes hang gliding?

   $$\frac{29}{90} = 0.322...$$

   c. a randomly selected student is a Senior?

   $$\frac{90}{400} = 0.225$$

   d. a randomly selected student who likes hang gliding is a senior?

   $$\frac{29}{147} = 0.197...$$

   e. a randomly selected student is a Junior?

   $$\frac{100}{400} = 0.25$$

   f. a randomly selected student who likes rock climbing is a Junior?

   $$\frac{39}{156} = 0.25$$

5. In the question "what is the probability that a randomly selected Senior likes hang gliding" the sample space is reduced from all students in the school to the group of __seniors_____.

   This probability is called **conditional probability**, and we use the notation P(HG | S). Read: the probability of HG given S.

6. Compute the following conditional probabilities using the numbers in the table.

a. P(HG | S) = $\dfrac{29}{90} = 0.322...$ P(S | HG) = $\dfrac{29}{147} = 0.197...$

b. P(RC | J) = $\dfrac{39}{100} = 0.39$ P(J | RC) = $\dfrac{39}{156} = 0.25$

7. How popular is hang gliding among Seniors, compared to the general school population?
Hang gliding is less popular among the Seniors than among the general school population.
$P(HG|S) = 0.322 < P(HG) = 0.3675$

It may be worth asking your students what they know about the popularity of hang gliding among non-seniors based on this conclusion.

8. How popular is rock climbing among Juniors, compared to the general school population? Contrast your answer with the answer to the previous question.
Rock climbing is equally popular among juniors and the general school population.
$P(RC|J) = 0.39 = P(RC)$

Can students make any statements about the popularity of rock climbing among non-juniors based on this result?

9. We say that two events are **independent**, if the occurrence of one does not affect the probability of the other. If the occurrence of one does affect the probability of the other, the two events are **dependent**. Tell whether the following event pairs are dependent or independent. Explain why.

a. HG and S Dependent. Knowing that you're talking to a senior makes it less likely that the student enjoys hang gliding.

b. RC and J Independent. Being a junior does not change the likelihood that a student enjoys rock climbing (see #8). Likewise, liking rock climbing doesn't make a student more (or less) likely to be a junior. $P(J|RC) = 0.25 = P(J)$

c. J and S Dependent. If a student is a junior, the probability that student is a senior drops to 0. HIGHLIGHT THIS. Students often confuse mutual exclusivity with independence.

Give the formal definition of independence using probabilities. Two events, A and B, are independent if ___ $P(A|B) = P(A)$ _____ and ___ $P(B|A) = P(B)$ _____ .

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10. How many students are Seniors <u>and</u> like hang gliding? 29

What is the probability that a randomly selected student is a Senior <u>and</u> likes hang gliding? Express this probability as a ratio. $\dfrac{29}{400}$

We could think about selecting these students in two steps. First, select all the Seniors. Then, from the group of Seniors select the students who like Hang gliding.
This is a good opportunity to use a tree to illustrate conditional probabilities, rather than a table.

The probability that a randomly selected student is a Senior <u>and</u> likes hang gliding is

P(S and HG) = $\dfrac{\#\ seniors}{\#\ total\_students} \bullet \dfrac{\#\ seniors\_who\_like\_hang\_gliding}{\#\ seniors} = \dfrac{}{400} \bullet \dfrac{29}{}$

Write the right-hand-side of the using probability notation.
$P(S \cap HG) = P(S) \bullet P(HG|S)$

11.   What is the probability that a randomly selected student is a Junior <u>and</u> likes rock climbing? Express this probability as a ratio.
$\dfrac{39}{400}$

Think about selecting these students in two steps. First, select all the Juniors. Then, from the group of Juniors, select those who like rock climbing. Express this probability as a product of two ratios, then write it using probability notation.

$$P(J \cap RC) = \frac{100}{400} \bullet \frac{39}{100} = P(J) \bullet P(RC|J)$$

What is the probability that a student likes rock climbing? Is this probability influenced by the student being a Junior (hint: what's the probability that a randomly selected Junior likes rock climbing)? Can P(J and RC) be expressed without conditional probabilities?
$P(RC|J) = 0.39 = P(RC)$, as we saw in # 8, so being a junior does not influence the probability that a student likes rock climbing. We can write P(J and RC) without conditional probabilities because these events are independent. $P(J \cap RC) = P(J) \bullet P(RC)$.

12.   Complete the description of the **multiplication rule**:

If A and B are events then P(A and B) = $P(A) \bullet P(B|A) = P(B) \bullet P(A|B)$
In the special case when the two events are independent, the rule becomes

P(A and B) = $P(A) \bullet P(B)$ because $P(B|A) = P(B)$ and $P(A|B) = P(A)$
It is likely worth noting to your students that the multiplication rule given above is a restatement of Bayes' Theorem.

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13. Write three examples (along with solutions) about the data in the table to share with your classmates. Be creative!

Many possibilities exist. Some will be more interesting than others.

Conditional Probability and Medical Testing

The following is a fictitious example. There is a deadly disease that affects about 3% of the population. If the disease is identified before symptoms are present, the patient will receive treatment and survive. A company that has been actively researching genetic testing announces a new test that they claim is 99% accurate in identifying the genetic marker that signals the disease. Let's examine this claim and its ramifications more closely. One option would be to create a tree diagram with the probabilities and to apply Bayes' Theorem. Another option, which will create a more intuitive way to check your answers, is to create a table of values.

1. Explain what you understand 99% accuracy to mean. Describe the ways that the test might misidentify individuals.

2. Let us assume that the population we are interested in is 1 million people. About 3% of this population has the genetic marker that signals the possibility of the deadly disease. Calculate the numbers in the table.

	Tests positive	Tests negative	Total
Has genetic marker			30,000
Doesn't have genetic marker			
Total			1,000,000

3. The test may turn out to be positive while the individual doesn't have the genetic marker. This case is called a **false positive**. Compute the probability that a randomly selected individual who tested positive does not have the marker.

4. The test may turn out to be negative while the individual does have the genetic marker. This case is called a **false negative**. Compute the probability that a randomly selected individual who tested negative <u>does</u> have the marker.

5. If you were a genetic counselor, what would you tell an individual who tested positive for the marker?

6. Are you surprised by the numbers you found? Is the test 99% percent accurate? Is there a contradiction between your findings and the company's claim?

7. Let's now build a model that will let us play with various aspects of this scenario. The easy way to do this is by creating a spreadsheet that will compute all of the desired values based on given inputs. The first question to ask is: what information might we vary? The logical choices are: test accuracy (originally 0.99), population size (originally 1,000,000), and disease incidence (originally 0.03). We'll start by recreating your original table as a way to double-check that your formulas are right. The basic setup is below.

8. Now we need to include the calculations, making sure to call cells B1 through B3 for values, rather than typing in the numbers directly. The first spaces to calculate are the row "total" cells on the right. How would you get 30,000 in cell G2? What about 970,000 in G3? Are you relying on your G2 calculation or performing an independent one?

   Formula in G2:

   Formula in G3:

9. Now that you have row totals, you can calculate the values within the table. Again, make sure to call cells rather than typing numbers:

Formula in E2:   Formula in F2:

Formula in E3:   Formula in F3:

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10. The column totals come next.

Formula in E4:   Formula in F4:

11. Last, we have the probabilities for false positives and false negative tests. You'll
    probably want to multiply by 100 within the cells.

Formula in B6:   Formula in B7:

Your spreadsheet should now look like this, matching the table you created by hand earlier.

	A	B	C	D	E	F	G	H
1	accuracy	0.99			pos_test	neg_test	total	
2	population	1000000		hasmarker	29700.	300.	30000.	
3	incidence	0.03		nomarker	9700.	960300.	970000.	
4				total	39400.	960600.		
5								
6	prob_false_pos	24.6192…						
7	prob_false_neg	0.03123…						
8								
9								
10								
11								

$B7 = \frac{f2}{f4} \cdot 100$

12. Vary the percent of the population with the genetic marker (cell B3) and look for patterns
    in the probabilities. List several value combinations [incidence, P(false positive), P(false
    negative)] and describe the patterns that you notice.

13. At what incidence level does a positive test result become less than 50% accurate? That
    is, for what incidence level is P(false positive) larger than 50%?

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14. Vary the population size (cell B2). Which cells change and which cells don't? How do you explain this?

15. You have built a numerical model and spotted patterns. Let's now build a theoretical model. Let x denote the fraction of the population that has the genetic marker (x is between 0 and 1, it is not a percentage.) Fill in the table in terms of x.

	Tests positive	Tests negative	Total
Has genetic marker			
Doesn't have genetic marker			
Total			1,000,000

16. Find the probability of false positives as a function of x. Simplify the function as much as you can.

17. Sketch this function here. Give all intercepts and any other key features. Give the practical domain and range of this function.

18. Does the shape of the graph support your answer about false positives in number 12?

19. An insurance company will not to pay for tests with false positive rates over 25%. Let's assume that there are different versions of the test based on the same research, with various accuracy levels. The more accurate the test, the more expensive it is. For the given accuracy levels, find the approximate value of the minimum percentage of the population that has the genetic marker that will have a false positive rate under 25%. You may use your spreadsheet to help check the values, but you should work out the answer for a 90% accurate test by setting up and solving a rational inequality as shown for the 99% accurate test.

Test accuracy	Min % of pop. with GM so that false positive rate is under 25%
99.5%	Spreadsheet:
99%	2.94%
97%	Spreadsheet:
95%	Spreadsheet:
90%	Calculated below:
85%	Spreadsheet:

$$\frac{1-x}{1+98x} \leq 0.25$$

$$\frac{1-x}{1+98x} - 0.25 \leq 0$$

$$\frac{1-x-0.25(1+98x)}{1+98x} \leq 0$$

$$1-x-0.25-24.5x \leq 0$$

$$0.75 - 25.5x \leq 0$$

$$0.75 \leq 25.5x$$

$$0.02941176 \leq x$$

20. Explain how you would use your observations if you had to decide which test to purchase for a particular community for which the percent of the population having the marker is known. For instance, consider 10% with the GM, then 15% with the GM.

Conditional Probability and Medical Testing

The following is a fictitious example. There is a deadly disease that affects about 3% of the population. If the disease is identified before symptoms are present, the patient will receive treatment and survive. A company that has been actively researching genetic testing announces a new test that they claim is 99% accurate in identifying the genetic marker that signals the disease. Let's examine this claim and its ramifications more closely. One option would be to create a tree diagram with the probabilities and to apply Bayes' Theorem. Another option, which will create a more intuitive way to check your answers, is to create a table of values.

21. Explain what you understand 99% accuracy to mean. Describe the ways that the test might misidentify individuals. Answers will vary.

If someone has the genetic marker, that person is 99% likely to test positive. If they don't have the marker, they're 99% likely to test negative. In terms of misidentifications, the test could come back negative when the person does have the marker or positive when they don't have it. In most cases for medical testing, we'd prefer the latter error to the former, since the consequences are generally less harmful.

The most common alternative explanation is that: someone who tests positive is 99% likely to have the genetic marker, and someone who tests negative is 99% likely not to have the genetic marker.

22. Let us assume that the population we are interested in is 1 million people. About 3% of this population has the genetic marker that signals the possibility of the deadly disease. Calculate the numbers in the table.

	Tests positive	Tests negative	Total
Has genetic marker	29,700	300	30,000
Doesn't have genetic marker	9,700	960,300	970,000
Total	39,400	960,600	1,000,000

23. The test may turn out to be positive while the individual doesn't have the genetic marker. This case is called a **false positive**. Compute the probability that a randomly selected individual who tested positive does not have the marker.

$$P\left(noGM\middle|+test\right)=\frac{9700}{39400}=0.246...$$

24. The test may turn out to be negative while the individual does have the genetic marker. This case is called a **false negative**. Compute the probability that a randomly selected individual who tested negative <u>does</u> have the marker.

$$P\left(GM\middle|-test\right)=\frac{300}{960600}=0.000312...$$

25. If you were a genetic counselor, what would you tell an individual who tested positive for the marker?

I would encourage follow-up testing, because almost 25% of those who test positive don't actually have the genetic marker.

26. Are you surprised by the numbers you found? Is the test 99% percent accurate? Is there a contradiction between your findings and the company's claim?

The test accurately classifies people with a 99% success rate. This is a different statement than a positive result meaning you're 99% likely to have the genetic marker. Because of the distribution of the genetic marker in the population, the inaccuracies are not evenly distributed between false positives and false negatives.

27. Let's now build a model that will let us play with various aspects of this scenario. The easy way to do this is by creating a spreadsheet that will compute all of the desired values based on given inputs. The first question to ask is: what information might we vary? The logical choices are: test accuracy (originally 0.99), population size (originally 1,000,000), and disease incidence (originally 0.03). We'll start by recreating your original table as a way to double-check that your formulas are right. The basic setup is below.

28. Now we need to include the calculations, making sure to call cells B1 through B3 for values, rather than typing in the numbers directly. The first spaces to calculate are the row "total" cells on the right. How would you get 30,000 in cell G2? What about 970,000 in G3? Are you relying on your G2 calculation or performing an independent one?

Formula in G2:   =b2*b3

Formula in G3:   =b2*(1-b3)   OR   =b2 – g2

29. Now that you have row totals, you can calculate the values within the table. Again, make sure to call cells rather than typing numbers:

Formula in E2:   =b1*g2   Formula in F2: =(1-b1)*g2 OR =g2 – e2

Formula in E3: =(1-b1)*g3 OR =g3-f3   Formula in F3:   =b1*g3

30. The column totals come next.

Formula in E4:   =e2 + e3   Formula in F4:   =f2 + f3

31. Last, we have the probabilities for false positives and false negative tests. You'll probably want to multiply by 100 within the cells.

Formula in B6: $= \dfrac{e3}{e4} * 100$ Formula in B7: $= \dfrac{f2}{f4} * 100$

Your spreadsheet should now look like this, matching the table you created by hand earlier.

	A	B	C	D	E	F	G	H
1	accuracy	0.99			pos_test	neg_test	total	
2	population	1000000		hasmarker	29700.	300.	30000.	
3	incidence	0.03		nomarker	9700.	960300.	970000.	
4				total	39400.	960600.		
5								
6	prob_false_pos	24.6192…						
7	prob_false_neg	0.03123…						
8								
9								
10								
11								

B7 $=\dfrac{f2}{f4} \cdot 100$

32. Vary the percent of the population with the genetic marker (cell B3) and look for patterns in the probabilities. List several value combinations [incidence, P(false positive), P(false negative)] and describe the patterns that you notice.

[0.05, 16.16, 0.05313], [0.10, 8.333, 0.1121], [0.25, 2.941, 0.3356], [0.01, 50, 0.0102], [0.001, 90.836, 0.00101]

As incidence goes down (a disease becomes more rare), the probability of false positives goes up, and the probability of false negatives goes down. Likewise, as incidence increases, the probability of false positives goes down.

33. At what incidence level does a positive test result become less than 50% accurate? That is, for what incidence level is P(false positive) larger than 50%?

For any incidence rate below 1% of the population, the probability of a false positive is above 50%.

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34. Vary the population size (cell B2). Which cells change and which cells don't? How do you explain this?

The numerical values in the table (E2, E3, E4, F2, F3, F4) all change, but the relationships between them don't. These probabilities in B6 and B7 (false positive, false negative) are based on proportions, while the numbers in the table depend on population size.

35. You have built a numerical model and spotted patterns. Let's now build a theoretical model. Let x denote the fraction of the population that has the genetic marker (x is between 0 and 1, it is not a percentage.) Fill in the table in terms of x.

	Tests positive	Tests negative	Total
Has genetic marker	$0.99 \cdot 10^6 \cdot x$	$0.01 \cdot 10^6 \cdot x$	$10^6 \cdot x$
Doesn't have genetic marker	$0.01 \cdot 10^6 (1-x)$	$0.99 \cdot 10^6 (1-x)$	$10^6 (1-x)$
Total	$10^6 (0.99x + 0.01(1-x))$	$10^6 (0.01x + 0.99(1-x))$	1,000,000

36. Find the probability of false positives as a function of x. Simplify the function as much as you can.

$$P(noGM \mid +test) = \frac{0.01 \cdot 10^6 (1-x)}{10^6 (0.99x + 0.01(1-x))} = \frac{1-x}{99x + 1(1-x)} = \frac{1-x}{1+98x}$$

37. Sketch this function here. Give all intercepts and any other key features. Give the practical domain and range of this function.

This is a rational function with intercepts at (0,1) and (1,0). It is worth talking about what these mean: if incidence is 0, meaning no one has the disease, 100% of the positives are false positives; if incidence is 1, meaning everyone has the disease, then 0% of the positives are false. The practical domain of the function is $0 \leq x \leq 1$ and the practical range is $0 \leq y \leq 1$. Outside of this, the function has a vertical asymptote at $x = -\frac{1}{98}$ and a horizontal asymptote at $y = -\frac{1}{98}$.

$$f3(x) = \frac{1-x}{1+98 \cdot x}$$

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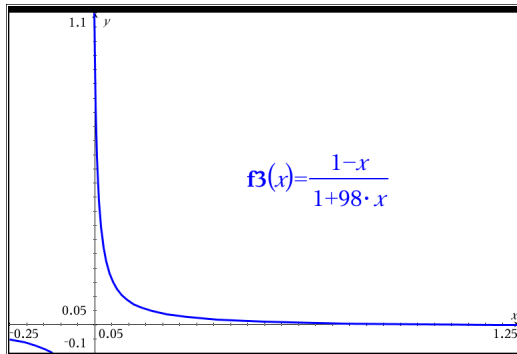
38. Does the shape of the graph support your answer about false positives in number 12?
Yes. As x gets closer to 0, y goes up and vice versa.

39. An insurance company will not to pay for tests with false positive rates over 25%. Let's assume that there are different versions of the test based on the same research, with various accuracy levels. The more accurate the test, the more expensive it is. For the given accuracy levels, find the approximate value of the minimum percentage of the population that has the genetic marker that will have a false positive rate under 25%. You may use your spreadsheet to help check the values, but you should work out the answer for a 90% accurate test by setting up and solving a rational inequality as shown for the 99% accurate test.

Test accuracy	Min % of pop. with GM so that false positive rate is under 25%
99.5%	Spreadsheet: 1.49%
99%	2.94%
97%	Spreadsheet: 8.5%
95%	Spreadsheet: 13.8%
90%	Calculated below: 25%
85%	Spreadsheet: 35%

$$\frac{1-x}{1+98x} \le 0.25$$
$$\frac{1-x}{1+98x} - 0.25 \le 0$$
$$\frac{1-x-0.25(1+98x)}{1+98x} \le 0$$
$$1-x-0.25-24.5x \le 0$$
$$0.75-25.5x \le 0$$
$$0.75 \le 25.5x$$
$$0.02941176 \le x$$

$$\frac{1-x}{1+8x} \le 0.25$$
$$\frac{1-x-0.25(1+8x)}{1+8x} \le 0$$
$$\frac{0.75-3x}{1+8x} \le 0$$
$$0.75 \le 3x$$
$$0.25 \le x$$

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40. Explain how you would use your observations if you had to decide which test to purchase for a particular community for which the percent of the population having the marker is known. For instance, consider 10% with the GM, then 15% with the GM.

If we know 10% of the population has the genetic marker, the 95% accurate test will not have a low enough rate of false positives (minimum percent of the population is 13.8%), so we'll need to purchase the 97% accurate test.
If 15% of the population has the genetic marker, the 95% accurate test will be fine (15>13.8).

451 Conditional Wrap-Up and Natural Frequencies

Let's return now to the breast cancer example in the initial question and use the ideas from our table method to build a quick way to estimate conditional probabilities. The approach is to use **natural frequencies**, and it can be far less reliant on complicated calculations. The idea is to pick a population size that makes calculations particularly straightforward for the given information. You may recall that changing the population size in your spreadsheet did not change the probabilities of false positives, so we can choose whatever is convenient for mental math.

Here's the given information again:
- The probability that a woman in this category has breast cancer is 1%.
- If a woman has breast cancer, the probability that she tests positive is 90%.
- If a woman does not have breast cancer, the probability that she tests positive anyway is 9%.

Let's choose a population of 1000 women, which should be easy to work with. After answering a few simple questions with calculations we can do in our heads, we should be able to answer the question of how likely a woman who has just tested positive is to actually have breast cancer. Since this is a method of quick estimation, you don't need perfect accuracy to be in the right ballpark.

1. How many of these 1000 women have breast cancer?

2. Of these, how many will test positive?

3. How many women who do not have breast cancer will test positive?

4. How many total positive tests does that make?

5. What is the approximate probability that a positive test is a true positive?

Wasn't that easier to do quickly in your head than using Bayes' Theorem (method shown below)?

$$P(cancer|positive) = \frac{P(cancer \& positive)}{P(positive)} = \frac{0.01*0.9}{0.01*0.9+0.99*0.09} = 0.0917$$

451 Conditional Wrap-Up and Natural Frequencies

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Let's choose a population of 1000 women, which should be easy to work with. After answering a few simple questions with calculations we can do in our heads, we should be able to answer the question of how likely a woman who has just tested positive is to actually have breast cancer. Since this is a method of quick estimation, you don't need perfect accuracy to be in the right ballpark.

6. How many of these 1000 women have breast cancer?
$0.01*1000 = 10$

7. Of these, how many will test positive?
$0.9*10 = 9$

8. How many women who do not have breast cancer will test positive?
$0.09*990 \approx 89$, but you don't have to do that in 1 step; 10% of 990 is 99, 1% of 990 is 9.9, so 9% is 99-9.9, or about 89.

9. How many total positive tests does that make?
$9 + 89 = 98$

10. What is the approximate probability that a positive test is a true positive?
$\frac{9}{98}$ which we can ballpark as somewhere between $\frac{1}{10}$ and $\frac{1}{11}$.

Wasn't that easier to do quickly in your head than using Bayes' Theorem (method shown below)?

$$P\left(cancer|positive\right) = \frac{P\left(cancer\ \&\ positive\right)}{P(positive)} = \frac{0.01*0.9}{0.01*0.9+0.99*0.09} = 0.0917$$

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Geometric Distribution Project

Introduction

The State Department has an urgent need to hire an Arabic speaker for a posting in the Middle East. Suppose 1/6 of all applicants speak the level of Arabic necessary for the job. We're going to study the question of how many candidates the State Department will have to interview before the meet *the first* candidate with sufficient skills in Arabic. We will first simulate this problem using dice. Then we will compute the theoretical probability distribution.

Experiment

1. Each student will roll one die repeatedly until a 6 is observed. The number of rolls it takes until the first 6 is rolled is the **outcome** of the experiment. Please repeat this experiment 20 times. On each trial, if it takes you *n* rolls to get your first 6, keep track of it by making a mark in the second row of the appropriate column. The number of times each outcome is observed is called the **frequency of the outcome**.

Outcome	1	2	3	4	5	6	7	8	9	10	11	≥ 12
Frequency												
Relative Frequency												

2. Compute the **relative frequency** of each outcome. That is, compute the ratio of the frequency to the number of times the experiment was carried out (in this case, 20). Write this information into the table.

3. Visualize the relative frequencies by creating a **histogram**. A histogram is a graphical display of the distribution of numerical data, typically by using bars of different heights.

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4. Collect data from the other groups in the classroom and compute the aggregate frequencies and relative frequencies for the whole class.

Outcome	1	2	3	4	5	6	7	8	9	10	11	≥ 12
Frequency												
Relative Frequency												

5. What effect does the additional data have on the relative frequencies? Create a new histogram showing the data from the class and compare it to your own histogram.

Theoretical distribution

6. Let's define success as rolling a 6 and failure as rolling any number other than 6. What is the probability of success (p) on any one roll? What's the probability of failure (q) on any one roll?

    $p=$ $q=$

7. What's the probability of rolling a 6 on the first try?

8. What's the probability of rolling a 6 *for the first time* on the second try?

9. What's the probability of rolling a 6 *for the first time* on the third try?

10. Write a formula for P(n), the probability of rolling a 6 *for the first time* on the nth try.

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11. Compute the theoretical values of P(*n*) for *n*= 1, 2, 3, …, 12 and complete the table below.

n	1	2	3	4	5	6	7	8	9	10	11	≥ 12
P(*n*)												

This is called the **probability mass function (pmf)** of the distribution.

12. Graph the histograms of relative frequencies from your data, the class data, and the pmf for the theoretical model on the same axes. Use a different color for each plot, labeling clearly which is which.

13. What happens to P(*n*) as *n* gets larger and larger? Explain why.

14. Why should P(1)+P(2)+P(3)+… equal 1? Use complete sentences.

15. Verify algebraically that P(1)+P(2)+P(3)+…=1. This will require you to remember your work with infinite series.

16. On average, how many times did you need to roll the die before getting the first 6?

17. On average, how many times did your class need to roll the die before getting the first 6?

18. On average, how many times does the theoretical model predict we need to roll the die before getting the first 6? This is the expected value (mathematical expectation) of the distribution.

19. Compare the sampling distribution obtained from your individual data with the theoretical distribution. Compare the sampling distribution obtained from the whole class data with the theoretical distribution. How could you make the discrepancies even smaller?

20. What did you learn as a result of doing this exploration that you did not know before?

Geometric Distribution Project

Introduction

The State Department has an urgent need to hire an Arabic speaker for a posting in the Middle East. Suppose 1/6 of all applicants speak the level of Arabic necessary for the job. We're going to study the question of how many candidates the State Department will have to interview before the meet ***the first*** candidate with sufficient skills in Arabic. We will first simulate this problem using dice. Then we will compute the theoretical probability distribution.

Experiment   You might simulate this using randint(1,6) on the calculator instead

21. Each student will roll one die repeatedly until a 6 is observed. The number of rolls it takes until the first 6 is rolled is the **outcome** of the experiment. Please repeat this experiment 20 times. On each trial, if it takes you *n* rolls to get your first 6, keep track of it by making a mark in the second row of the appropriate column. The number of times each outcome is observed is called the **frequency of the outcome**.

Outcome	1	2	3	4	5	6	7	8	9	10	11	≥ 12
Frequency												
Relative Frequency												

Answers will vary. Some will be further from the theoretical distribution than others.

22. Compute the **relative frequency** of each outcome. That is, compute the ratio of the frequency to the number of times the experiment was carried out (in this case, 20). Write this information into the table.

23. Visualize the relative frequencies by creating a **histogram**. A histogram is a graphical display of the distribution of numerical data, typically by using bars of different heights.

Again, answers will vary. Look to make sure axes are scaled and labeled appropriately.

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24. Collect data from the other groups in the classroom and compute the aggregate frequencies and relative frequencies for the whole class.

Outcome	1	2	3	4	5	6	7	8	9	10	11	≥12
Frequency												
Relative Frequency												

Answers will still vary, but the numbers for the whole class should be closer to the theoretical probabilities.

25. What effect does the additional data have on the relative frequencies? Create a new histogram showing the data from the class and compare it to your own histogram.

Answers will vary depending on how close the individual numbers were to the class as a whole.

Theoretical distribution

26. Let's define success as rolling a 6 and failure as rolling any number other than 6. What is the probability of success (p) on any one roll? What's the probability of failure (q) on any one roll?

$p = \dfrac{1}{6}$ $\qquad\qquad q = \dfrac{5}{6}$

27. What's the probability of rolling a 6 on the first try?

$\dfrac{1}{6} = 0.167$

28. What's the probability of rolling a 6 *for the first time* on the second try?

$\dfrac{5}{6} \cdot \dfrac{1}{6} = \dfrac{5}{36} = 0.139$

29. What's the probability of rolling a 6 *for the first time* on the third try?

$\left(\dfrac{5}{6}\right)^2 \cdot \dfrac{1}{6} = \dfrac{25}{216} = 0.116$

30. Write a formula for P(n), the probability of rolling a 6 *for the first time* on the nth try.

$P(n) = \left(\dfrac{5}{6}\right)^{n-1} \cdot \dfrac{1}{6} = \dfrac{5^{n-1}}{6^n}$

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31. Compute the theoretical values of P(*n*) for *n*= 1, 2, 3, …, 12 and complete the table below.

n	1	2	3	4	5	6	7	8	9	10	11	12
P(*n*)	0.167	0.139	0.116	0.096	0.080	0.067	0.056	0.047	0.039	0.032	0.027	0.022

This is called the **probability mass function (pmf)** of the distribution.

32. Graph the histograms of relative frequencies from your data, the class data, and the pmf for the theoretical model on the same axes. Use a different color for each plot, labeling clearly which is which.

33. What happens to P(*n*) as *n* gets larger and larger? Explain why.

P(*n*) gets smaller as *n* gets larger. $P(n) = \frac{5}{6}P(n-1)$, so each successive value is 5/6 as large as the previous value.

34. Why should P(1)+P(2)+P(3)+… equal 1? Use complete sentences.
Eventually you will roll a 6. Thus the sum of these mutually exclusive events must be 1, because one of the events is sure to happen.

35. Verify algebraically that P(1)+P(2)+P(3)+…=1. This will require you to remember your work with infinite series.

$$P(1)+P(2)+P(3)+... = \frac{1}{6} + \frac{5}{6}\cdot\frac{1}{6} + \left(\frac{5}{6}\right)^2\frac{1}{6} + ... = \frac{1}{6}\left(1 + \frac{5}{6} + \left(\frac{5}{6}\right)^2 + \left(\frac{5}{6}\right)^3 + ...\right)$$ The quantity on

the right in this expression is an infinite geometric series with $r = \frac{5}{6}$, so

$$S_\infty = \frac{a_1}{1-r} = \frac{1}{1-\frac{5}{6}} = \frac{1}{1/6} = 6$$ and the overall sum then is $\frac{1}{6}\cdot 6 = 1$.

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36. On average, how many times did you need to roll the die before getting the first 6?
Answers will vary, but they should be of the form:
1*(1st relative frequency) + 2*(2nd relative frequency) + 3*(3rd relative frequency) +…=close to 6

37. On average, how many times did your class need to roll the die before getting the first 6?
This time, all in the class should have the same values, in the same form as above.

38. On average, how many times does the theoretical model predict we need to roll the die before getting the first 6? This is the expected value (mathematical expectation) of the distribution.

Let $S = 1 + \dfrac{5}{6} + \left(\dfrac{5}{6}\right)^2 + \left(\dfrac{5}{6}\right)^3 + ... = 6$.

$$EV = 1 \bullet \dfrac{1}{6} + 2\left(\dfrac{5}{6}\right)\dfrac{1}{6} + 3\left(\dfrac{5}{6}\right)^2 \dfrac{1}{6} + 4\left(\dfrac{5}{6}\right)^3 \dfrac{1}{6} + ...$$

$$= \dfrac{1}{6}\left[\left(1 + \dfrac{5}{6} + \left(\dfrac{5}{6}\right)^2 + \left(\dfrac{5}{6}\right)^3 + ...\right) + \left(\dfrac{5}{6} + \left(\dfrac{5}{6}\right)^2 + \left(\dfrac{5}{6}\right)^3 + ...\right) + \left(\left(\dfrac{5}{6}\right)^2 + \left(\dfrac{5}{6}\right)^3 + ...\right) + ...\right]$$

$$= \dfrac{1}{6}\left[S + \dfrac{5}{6}S + \left(\dfrac{5}{6}\right)^2 S + ...\right] = \dfrac{1}{6}S\left(1 + \dfrac{5}{6} + \left(\dfrac{5}{6}\right)^2 + \left(\dfrac{5}{6}\right)^3 + ...\right) = \dfrac{1}{6}S^2 = \dfrac{1}{6} * 36 = 6$$

39. Compare the sampling distribution obtained from your individual data with the theoretical distribution. Compare the sampling distribution obtained from the whole class data with the theoretical distribution. How could you make the discrepancies even smaller?
The more trials we do, the closer the sampling distribution will be to the theoretical distribution. Interestingly, though, the expected values obtained by the sampling distributions are pretty good, even with a limited number of trials.

40. What did you learn as a result of doing this exploration that you did not know before?
Answers will vary.

APPENDIX F: Unit 7 Activities and Solutions

The Predator/Prey activity can be found here:
http://www.stewartmath.com/dp/DP-CH-5-a.pdf

The Movie Seating activity can be found here:
http://www.stewartmath.com/dp/DP-CH-7-a.pdf

MA452 Predator/Prey Solutions

1. For the hares, peak height is 3500 and trough height is 500, so amplitude is
$\dfrac{3500-500}{2} = 1500$. The midline is halfway between values at 2000 hares. The curve starts on
the midline going up, so it is a positive sine curve with no horizontal shift. With a period of 120
weeks, $B = \dfrac{2\pi}{120} = \dfrac{\pi}{60}$. This gives us the equation $H(t) = 1500\sin\left(\dfrac{\pi}{60}t\right) + 2000$ to describe the
population of hares.

For the lynx, there is a trough at 400 animals and a peak at 2000 animals, so the amplitude is
800. The midline is at 1200 lynx. The period is the same 120 weeks (troughs at 10 and 130
weeks), so the population of lynx can be described by $L(t) = -800\cos\left(\dfrac{\pi}{60}(t-10)\right) + 1200$.

2. Using the calculator to add the 2 functions and analyze the resulting graph, we have peaks at
$(40.735, 4500)$ and $(160.735, 4500)$ and a trough at $(100.735, 1900)$. The period is the same 120
weeks, the amplitude is 1300, the midline is 3200, and we can find a horizontal shift right to the
first peak of 40.735. The function for the total number of animals, then, is
$P(t) = 1300\cos\left(\dfrac{\pi}{60}(t-40.735)\right) + 3200$. The midline is the sum of the average values of the 2
populations taken individually, and the period is the same as the shared period of the individual
populations, unsurprisingly.
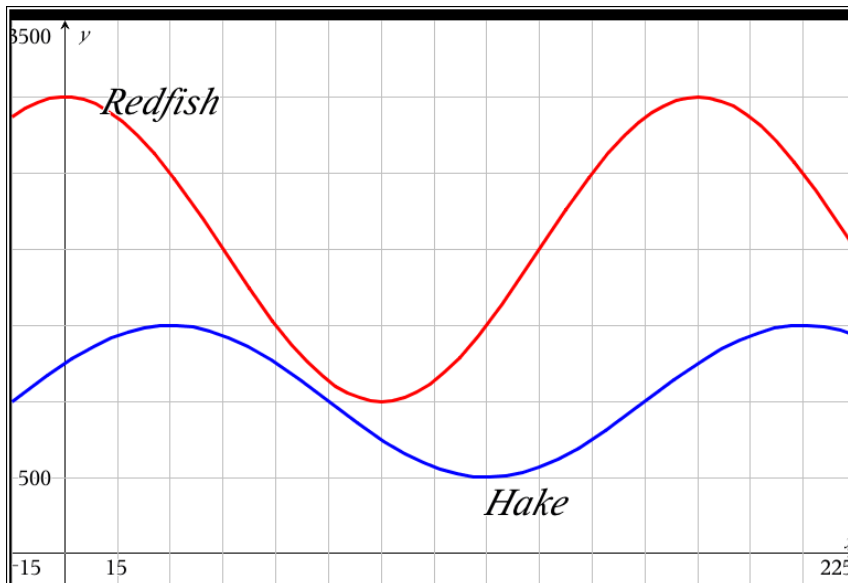
3. (a)
$$\text{Re}\,dfish(t) = 1000\cos\left(\dfrac{\pi}{90}t\right) + 2000$$
(b)
$$Hake(t) = 500\cos\left(\dfrac{\pi}{90}(t-30)\right) + 1000$$

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4. This asks students to draw sinusoidal graphs in which amplitude decreases over time for both predators and prey. You might want to demonstrate damped oscillation, or you may treat this as an extension for an honors group.

Both of these examples (hares and lynx and redfish and hake) lend themselves nicely to introducing the idea of phase plane (hares on one axis, lynx on the other) and parametric graphs. While you may not go into great detail about either, it is worthwhile to show one of these and utilize the trace feature to show ordered pairs of (hares, lynx) and progression around in the phase plane. This is a preview of work to be done with slope fields later in their mathematical careers.

The graph below shows lynx on the *x*-axis and hares on the *y*-axis.

$x2, y2$: $(401 , 2.68\text{E}{+}3)$, $t = 9$
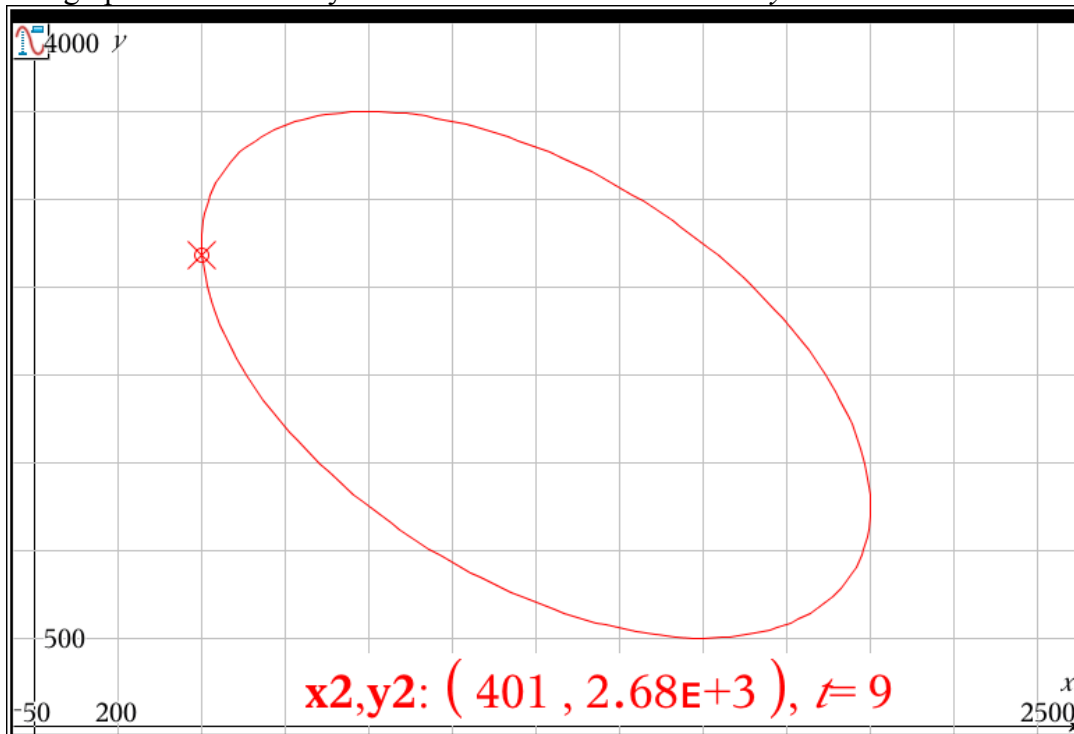
MA452 Where to Sit at the Movies Solutions

1. a) It is often easier to deal with right triangles, so we'll turn the given picture into one with right triangles.

22 ft

6 ft

α

x
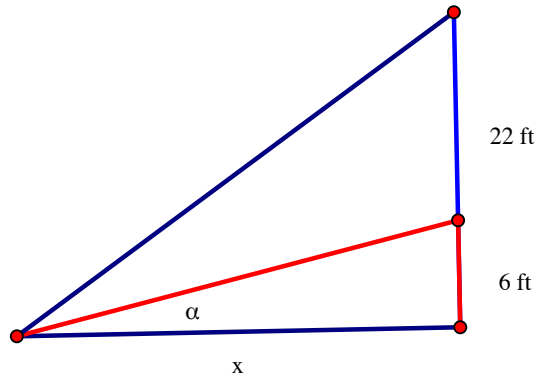
If we let β be the angle in the large triangle with legs of x and 28, then $\theta = \beta - \alpha$. We also have that $\tan\beta = \dfrac{28}{x}, \tan\alpha = \dfrac{6}{x}$. Using inverse trig, we see that $\beta = \tan^{-1}\left(\dfrac{28}{x}\right), \alpha = \tan^{-1}\left(\dfrac{6}{x}\right)$.

$$\tan\theta = \tan(\beta - \alpha) = \frac{\tan\beta - \tan\alpha}{1 + \tan\beta\tan\alpha}$$

b) $$= \frac{\dfrac{28}{x} - \dfrac{6}{x}}{1 + \dfrac{28}{x}\dfrac{6}{x}} \cdot \frac{x^2}{x^2} = \frac{28x - 6x}{x^2 + 168} = \frac{22x}{x^2 + 168}$$

$$\theta = \tan^{-1}\left(\frac{22x}{x^2 + 168}\right)$$

$\dfrac{\pi}{2}+1$ y

$\mathbf{f1}(x)=\tan^{-1}\left(\dfrac{22\cdot x}{x^2+168}\right)$

$(13, 0.704)$

$\dfrac{\pi}{12}$

-2 5 60

-1
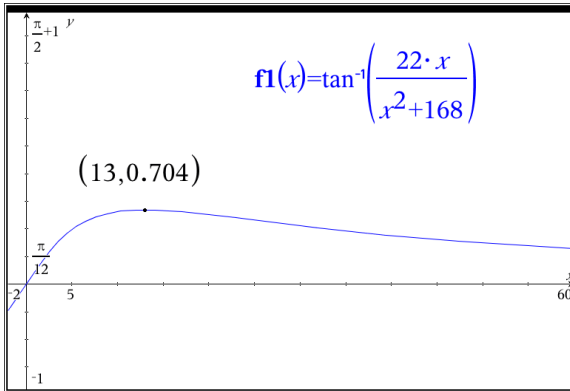
c) The max value is when x=13 feet from the screen, which puts you in the third row.

2. a) The easy part of this problem is to use the law of cosines. Let a and b be the 2 sides of the triangle that form angle θ, with the screen as the third side. Playing with the law of cosines, we

$$22^2 = a^2 + b^2 - 2ab\cos\theta$$
$$2ab\cos\theta = a^2 + b^2 - 484$$

have that $\cos\theta = \dfrac{a^2 + b^2 - 484}{2ab}$

$$\theta = \cos^{-1}\left(\frac{a^2 + b^2 - 484}{2ab}\right)$$

The harder part is to figure out how a and b relate to x. To do this, we will again utilize right triangles. Looking at the triangle that forms the base of the seats, we see that the horizontal distance from the wall to the viewer is $7 + x\cos\alpha$. We use this horizontal distance as one leg of both right triangles formed from the viewer's eye.

a

22+10-(4+xsinα)

7+xcosα

b

4+xsinα-10
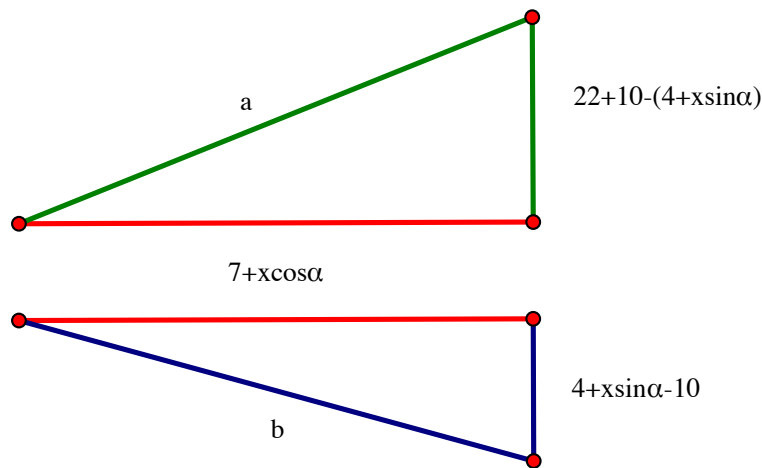
We get a^2 and b^2 in terms of x as shown in the question simply by using the Pythagorean Theorem.

b) The graph is shown below for an angle of $\alpha = 25°$.

$$f2(x) = \cos^{-1}\left(\frac{a^2 + b^2 - 484}{2 \cdot a \cdot b}\right)$$

$(9.3, 0.867)$
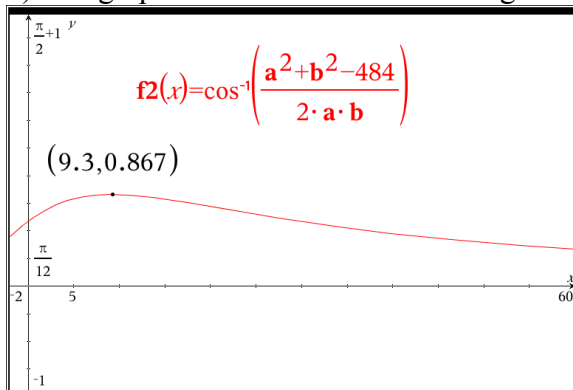
The goal is to be 9.3 feet up the slope. This corresponds to a horizontal distance of 8.429 feet. While we can't hit this perfectly, given the fixed seating in the theater (3 feet horizontally between rows), our best choice is to sit in the 4th row.

THE CHOCOLATE FACTORY

In many mechanical devices, circular motion is converted to linear motion or linear motion is converted to circular motion to apply energy as required.

Situation

In a chocolate factory in Ghirardelli Square in San Francisco, a wheel that pushes a blade back and forth across the bottom of the vat, as illustrated below, drives the equipment that stirs the milk chocolate mixture. The device is situated so that point T is on the same level as the center of the wheel, point C.
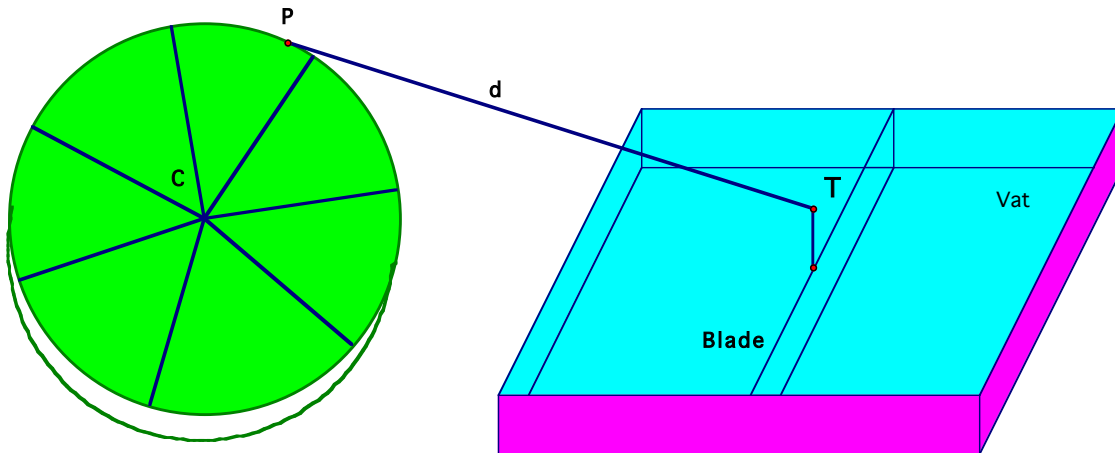

Problem

(a) Design a vat of appropriate length and place it the proper distance from the center of the wheel, given that the wheel has a radius of 2 meters and the rod attached to the stirring blade is 5 meters long. Do this now, making sure you have a good overhead drawing of the vat and wheel and the important distances.

(b) Determine the location of the blade in the vat for any angle of rotation of the wheel. This will take further investigation (see below).

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PART I. INITIAL EXPLORATION

1. On the diagrams below, sketch the stirring machine at different angles of rotation for the wheel. Measure all angles in standard position. The stirring blade is attached at the level of the center of the wheel. Investigate $0°$ (no drawing needed), $30°$, $180°$, $225°$
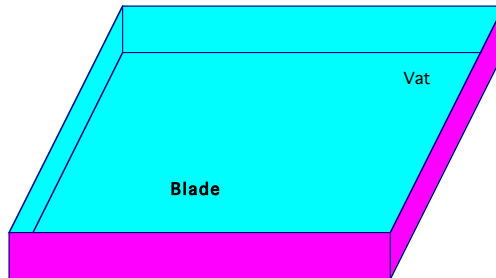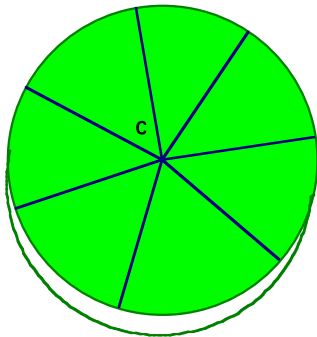
Angle Measure _____

Angle Measure _____

Angle Measure _____

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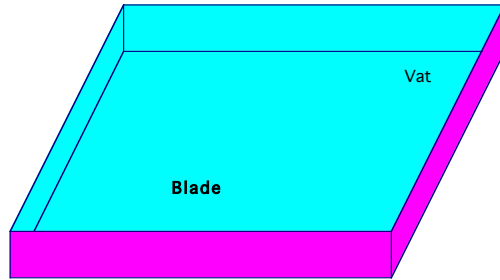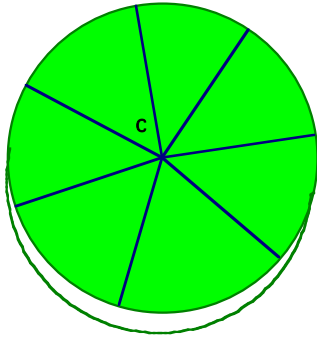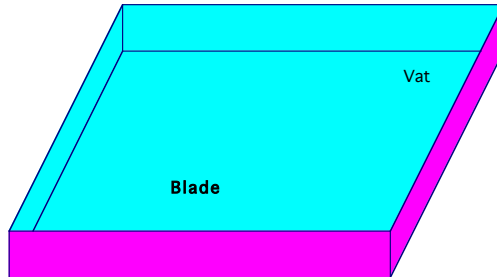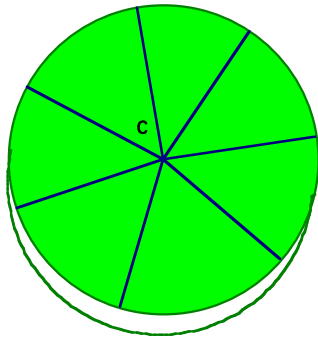
PART II. START TO GENERALIZE

Do all work for this part on a separate page. Continue to assume that CP=2m and PT=5m.

2. Describe the horizontal distance of point P to the left or right of the center of the wheel (x in the drawing below) as a function of the angle of the rotation. Assume that the wheel is rotating counterclockwise and the angle of rotation, θ, is the angle between the horizontal ray through the center of the wheel directed to the right (CT is a segment of this) and the radius CP.
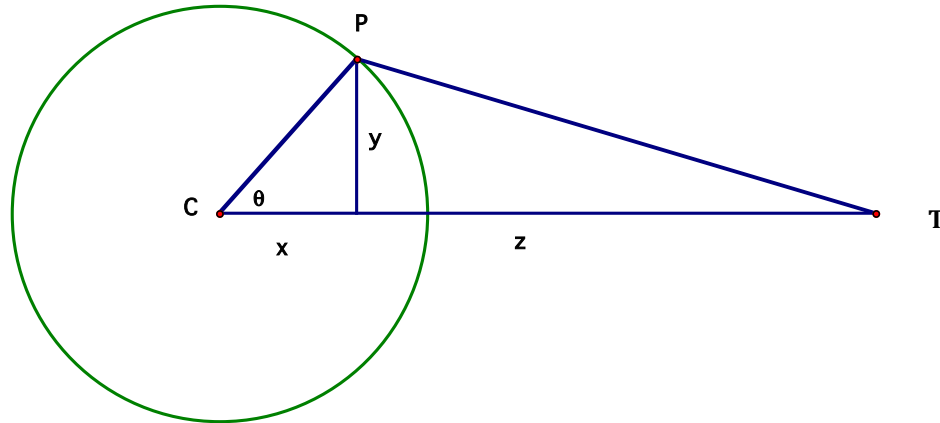

You may also assume that x is perpendicular to y.

3. Describe the horizontal distance from point P on the wheel to the stirrer blade as a function only of the angle of rotation (in other words, find z in terms of θ). This will take more work than number 2; be careful with the algebra.

4. How far is the stirrer blade (at point T) from the center of the wheel for each angle of rotation? Write a function for the distance ($x+z$) in terms of the angle (your result should be in the form d(θ)=...).

5. Verify that your answer in problem 4 is correct by evaluating d(0º), d(30º), d(180º) and d(225º) and comparing the results with your answers in Part I.

PART III. GENERAL CASE

Assume that the wheel is rotating counterclockwise and the angle of rotation, θ, is the angle between the horizontal ray through the center of the wheel directed to the right and the radius CP. If the stirring blade is *s* meters long and the radius of the wheel is *r* meters, how far is the stirrer blade (at point T) from the center of the wheel for each angle of rotation? Write a function for the distance CT in terms of *s*, *r*, and θ. Once you have the function, play with it a little; solve for s^2.
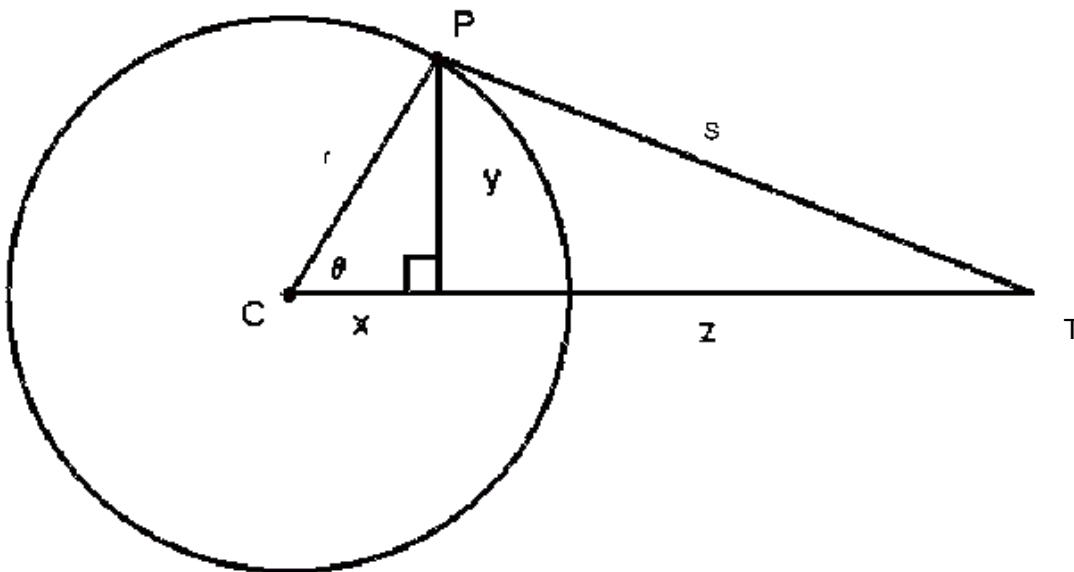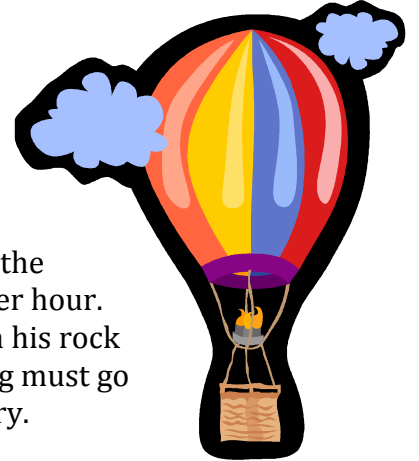

Solve for s²:

$$CT = r\cos\theta + \sqrt{s^2 - r^2 \sin^2 \theta}$$

APPENDIX G: Unit 8 Activities and Solutions

Wile E Coyote is once again after a good meal of Road Runner and has devised an ingenious plan to capture the bird. Using his handy ACME catalog, he has ordered a bicycle with a special launcher built on the back. With this launcher, he will 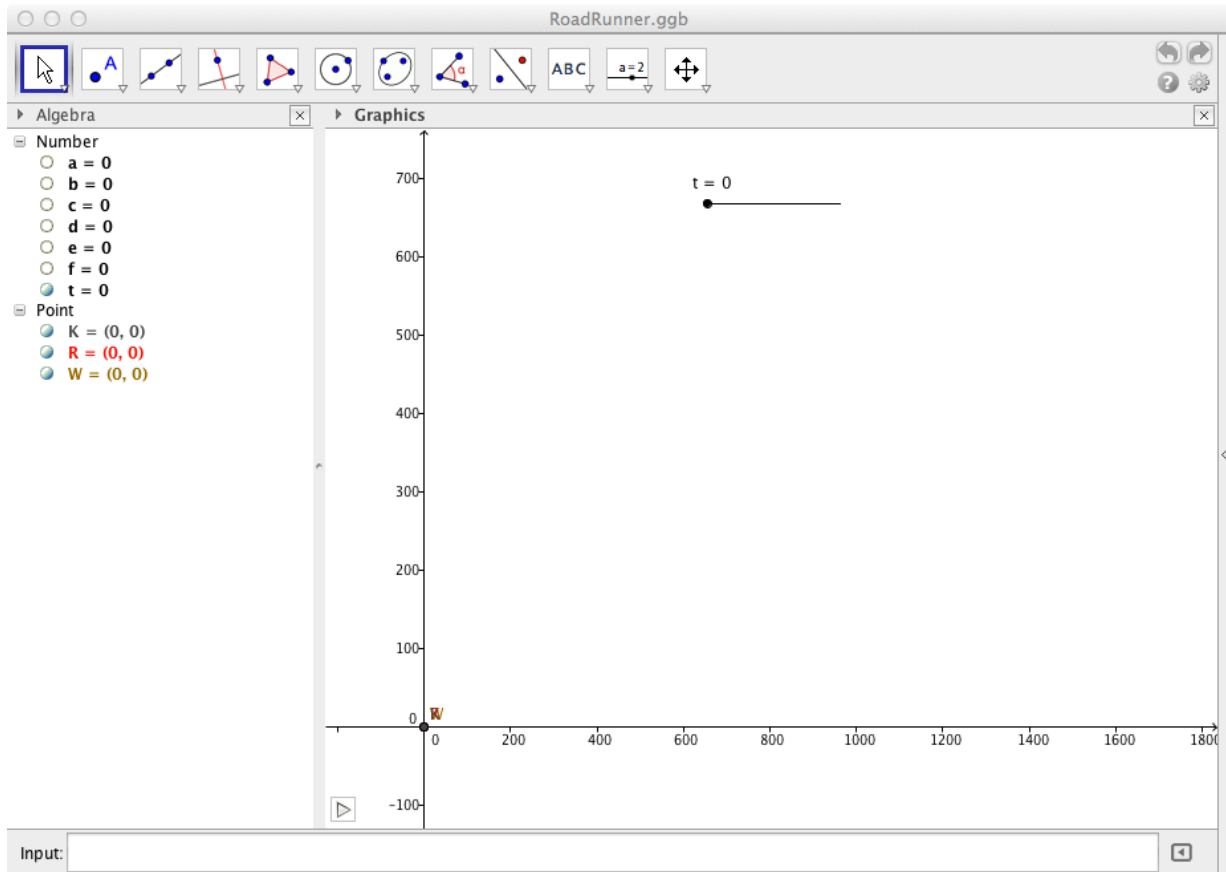hurl a rock at Road Runner as the bird floats by in a hot air balloon at an altitude of 200 feet above the coyote and a speed of 10 miles per hour. According to his plans, Wile E will ride his bike towards the spot directly under Road Runner's balloon at 15 miles per hour. When he is exactly 400 feet from this spot he will launch his rock with an upward speed of 130 feet per second. Everything must go precisely according to plan or Wile E will go home hungry.

1. What do you think of Wile E's plan?

2. Write equations for Wile E (on his bike), the Road Runner (in his balloon), and the rock that will mathematically model the motions. Make sure your units are consistent. Once you have equations, open the GeoGebra file "RoadRunner." Double-click on "a" to enter the *x* equation for Wile E, "b" for his *y* equation. Double-click on "c" and "d" for Road Runner's *x* and *y* equations, and "e" and "f" for the rock's *x* and *y* equations. To watch the scenario unfold, click the play arrow in the lower left of the window. Report your findings.
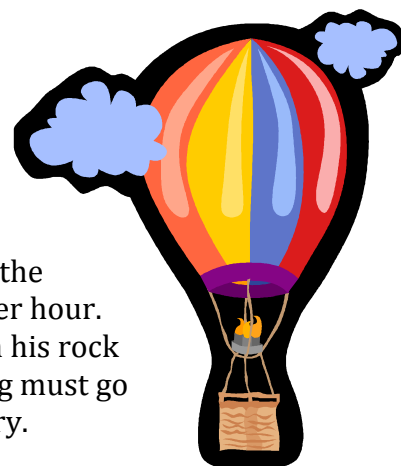
3. Did you remember to account for the wind that is causing Road Runner to move at 10 miles per hour? Incorporate this idea into your equations for the rock and rerun the simulation. Report your findings.

4. Now that you are feeling very sad for Wile E, you may be surprised to hear that Wile E didn't read all of the directions. He can increase the upward velocity of the rock by turning a knob on the launcher. The maximum upward velocity is 200 feet per second. Could Wile E change only this parameter (f) and hit Road Runner?

5. Another part of the directions says that Wile E can change the angle of the launcher by as much as 45 degrees from the vertical. Can you change only the angle of the launcher (keeping the initial velocity at 130 feet per second) to help Wile E get the Road Runner? To do this, you'll need trigonometry to break the rock's velocity into its x and y components. To ensure appropriate calculations in GeoGebra, use the $°$ symbol in your equations (find this by clicking on α at the right of the entry line).

6. Can you alter both speed at launch and tilt of the launcher to help Wile E get the Road Runner? Give the equations used in several trials and what the result of each trial was.

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Wile E Coyote is once again after a good meal of Road Runner and has devised an ingenious plan to capture the bird. Using his handy ACME catalog, he has ordered a bicycle with a special launcher built on the back. With this launcher, he will hurl a rock at Road Runner as the bird floats by in a hot air balloon at an altitude of 200 feet above the coyote and a speed of 10 miles per hour. According to his plans, Wile E will ride his bike towards the spot directly under Road Runner's balloon at 15 miles per hour. When he is exactly 400 feet from this spot he will launch his rock with an upward speed of 130 feet per second. Everything must go precisely according to plan or Wile E will go home hungry.

1. What do you think of Wile E's plan?
   His plans never work. This one, too, will fail in some spectacular way.

2. Write equations for Wile E (on his bike), the Road Runner (in his balloon), and the rock that will mathematically model the motions. Make sure your units are consistent. Once you have equations, open the GeoGebra file "RoadRunner." Double-click on "a" to enter the x equation for Wile E, "b" for his y equation. Double-click on "c" and "d" for Road Runner's x and y equations, and "e" and "f" for the rock's x and y equations. To watch the scenario unfold, click the play arrow in the lower left of the window. Report your findings.

   $10\,mph = 14\frac{2}{3}\,fps \qquad 15\,mph = 22\,fps$

   For Wile E (point W): $\begin{array}{l} a = 22t \\ b = 0 \end{array}$

   For Road Runner (point R): $\begin{array}{l} c = 14\frac{2}{3}t + 400 \\ d = 200 \end{array}$

   For the rock (point K): $\begin{array}{l} e = 22t \\ f = -16t^2 + 130t \end{array}$

   The rock misses the Road Runner and comes back down to hit Wile E on the head.

3. Did you remember to account for the wind that is causing Road Runner to move at 10 miles per hour? Incorporate this idea into your equations for the rock and rerun the simulation. Report your findings.

   Nope, forgot the wind. We'll need to adjust so that $e = 22t + 14\frac{2}{3}t = 36\frac{2}{3}t$. The rock still misses the Road Runner, but at least it doesn't hit Wile E anymore.

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4. Now that you are feeling very sad for Wile E, you may be surprised to hear that Wile E didn't read all of the directions. He can increase the upward velocity of the rock by turning a knob on the launcher. The maximum upward velocity is 200 feet per second. Could Wile E change only this parameter (f) and hit Road Runner?

   No, even with the wind, the difference in speeds is not large enough for the rock to close the initial 400-foot horizontal gap. Most students will find this by plugging in the values and checking to see what happens.

5. Another part of the directions says that Wile E can change the angle of the launcher by as much as 45 degrees from the vertical. Can you change only the angle of the launcher (keeping the initial velocity at 130 feet per second) to help Wile E get the Road Runner? To do this, you'll need trigonometry to break the rock's velocity into its x and y components. To ensure appropriate calculations in GeoGebra, use the ° symbol in your equations (find this by clicking on α at the right of the entry line).

   No; to get the right horizontal distance, you give up too much height.

   $$e = 36\frac{2}{3}t + 130t\cos(\theta)$$
   $$f = -16t^2 + 130t\sin(\theta)$$

6. Can you alter both speed at launch and tilt of the launcher to help Wile E get the Road Runner? Give the equations used in several trials and what the result of each trial was.

   Multiple combinations are possible.

MA452 Lines 3 Ways

As you work through this sheet, you will be reminded of work you have already done on standard equations of lines and vectors and their components. You will be introduced to vector equations for lines and parametric equations for lines.

1. Find the point-slope equation of the line containing points $A(1,2)$ and $B(5,8)$.

One potential disadvantage of this form of a line is that it tells you nothing about the direction or speed of travel if an object is moving on this path.

2. Find the *x*- and *y*-components of the vector that goes from point *A* to point *B*.

You will recall that a vector gives both direction and magnitude, and that multiplying a vector by a positive scalar (constant) changes the magnitude but not the direction. For example, $2\vec{v}$ goes in the same direction as $\vec{v}$ but twice as far. You may also recall that multiplying by a negative scalar involves travel in exactly the opposite direction. That is, if $\vec{w}$ gives the vector from point A to point B, then $-\vec{w}$ gives the vector from B to A. Let's return to our initial example, with points $A(1,2)$ and $B(5,8)$. We calculated (in problem 2) the component form of vector $\overrightarrow{AB}$. If we start at point *A* and move one $\overrightarrow{AB}$, we will be at point *B*.

3. What are the coordinates of the point we will end on if we start at *A* and move:

(a) $2\overrightarrow{AB}$ (b) $5\overrightarrow{AB}$ (c) $-\overrightarrow{AB}$ (d) $-\frac{1}{2}\overrightarrow{AB}$

You will note that all 4 of these points fall on our initial line from problem 1. If we take an arbitrary number, say *t,* copies of $\overrightarrow{AB}$, we generate every point on the line containing *A* and *B*. It is this idea that allows us to generate the **vector equation for a line**, which gives a formula in terms of the *x*- and *y*-coordinates of any point on the line:

$(x,y)=(x_1,y_1)+t\langle x_c,y_c\rangle$

where (x_1,y_1) are the coordinates of the initial point and $\langle x_c,y_c\rangle$ are the components of the initial vector. In our example, the vector equation of the line from $A(1,2)$ to $B(5,8)$ is

$(x,y)=(1,2)+t\langle 4,6\rangle$

You might want to confirm that if you let t equal 2, 5, -1, and $-\dfrac{1}{2}$ you will get the coordinates of the points in problem 3, and that each of these points satisfies the equation you found in problem 1 for the line containing A and B.

4. Find the vector equation of the line from $B(5,8)$ to $A(1,2)$.

5. Find the vector equation of the line from $C(-2,5)$ to $D(1,-3)$.

You will, no doubt, recall the challenges of solving navigation problems solely by trigonometry and the laws of sines and cosines (a well-drawn picture was truly worth 1000 words). You may also recall that these problems got much easier when we broke them down into x- and y-components and worked with each separately. There are many contexts in which it is easier to split x and y apart and consider them somewhat separately (think, for example, of an object flying initially horizontally through the air, subject to gravity and wind resistance in the vertical direction and wind resistance in the horizontal direction). We begin that work, studying x and y as functions of the **parameter** t, now.

The **parametric equations for a line** take the vector equation for that line and split it into its constituent parts. To get the parametric equations for the line from $A(1,2)$ to $B(5,8)$, we start with $(x,y) = (1,2) + t\langle 4,6 \rangle$ and break it into the x piece and the y piece separately:
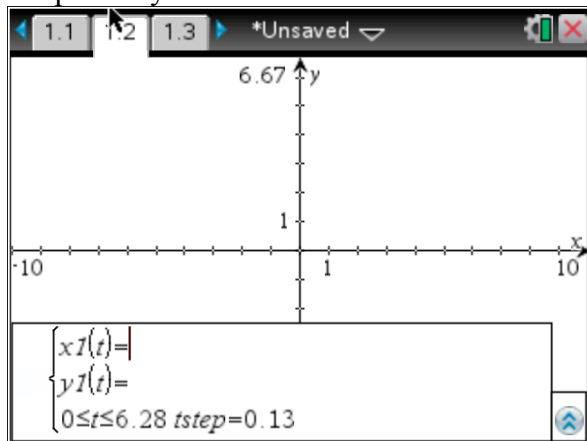$x = 1 + 4t$
$y = 2 + 6t$
Notice that when $t=1$, $x=5$ and $y=8$, as desired.

6. Find the parametric equations for the lines given in problems 4 and 5.

7. Find vector and parametric equations for the line that goes from $E(-3,4)$ to $F(7,6)$.

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8. Find vector and parametric equations for the line that goes from $P(2,5)$ to $Q(-2,-1)$.
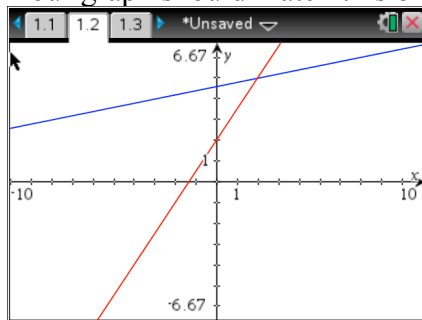
Your calculator can graph parametric equations. Open a graphs page, then choose Menu -> 3: Graph Entry/Edit -> 3: Parametric. Your screen should now look like this:

There are a couple important features to notice here. First, *x1(t)=* and *y1(t)=* have replaced the *f1(x)=* that you find on a standard graphs page. Second, the bottom line is crucial to what you will see: **you have control over what values *t* will take on.** As you can see, the default is for *t* to vary between 0 and 2π. While this foreshadows the uses for parametric equations that are coming soon, it is not helpful for viewing lines. You'll want to include negative values of *t*, and, depending on your viewing window and the slope of your line, you may need to go further than 2π on either side of 0.

9. Use your Nspire to graph the parametric equations for the lines you found in problems 7 and 8. Your graph should match this one:

As you can see, the lines intersect. This guarantees a "path intersection," but we can now concern ourselves with the idea of a "particle collision." The question is now whether a particle traveling according to the first set of equations and one traveling according to the second set will actually collide, or whether they will pass the point of intersections at different times. There are a number of ways to find an answer to this question, but the easiest involves finding the time at which the *x*-coordinates coincide and the time at which the *y*-coordinates coincide and seeing if they are the same:

$$-3+10t = 2-4t \qquad\qquad 4+2t = 5-6t$$

For *x:* $14t = 5$ For *y:* $8t = 1$ Since the simultaneous *x* and

$$t = \frac{5}{14} \qquad\qquad\qquad t = \frac{1}{8}$$

y occupations happen at different times, these 2 particles do not collide.

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MA452 Lines 3 Ways

As you work through this sheet, you will be reminded of work you have already done on standard equations of lines and vectors and their components. You will be introduced to vector equations for lines, and parametric equations for lines.

1. Find the point-slope equation of the line containing points $A(1,2)$ and $B(5,8)$.

Slope: $m = \dfrac{8-2}{5-1} = \dfrac{6}{4} = \dfrac{3}{2}$

Equation: $y - 2 = \dfrac{3}{2}(x-1)$ or $y - 8 = \dfrac{3}{2}(x-5)$

One potential disadvantage of this form of a line is that it tells you nothing about the direction or speed of travel if an object is moving on this path.

2. Find the *x*- and *y*-components of the vector that goes from point *A* to point *B*.

$\langle 4,6 \rangle$

You will recall that a vector gives both direction and magnitude, and that multiplying a vector by a positive scalar (constant) changes the magnitude but not the direction. For example, $2\vec{v}$ goes in the same direction as $\vec{v}$ but twice as far. You may also recall that multiplying by a negative scalar involves travel in exactly the opposite direction. That is, if $\vec{w}$ gives the vector from point A to point B, then $-\vec{w}$ gives the vector from B to A. Let's return to our initial example, with points $A(1,2)$ and $B(5,8)$. We calculated (in problem 2) the component form of vector $\overrightarrow{AB}$. If we start at point *A* and move one $\overrightarrow{AB}$, we will be at point *B*.

3. What are the coordinates of the point we will end on if we start at *A* and move:

(a) $2\overrightarrow{AB}$ (b) $5\overrightarrow{AB}$ (c) $-\overrightarrow{AB}$ (d) $-\dfrac{1}{2}\overrightarrow{AB}$

(9,12)   (21,32)   (–3,–4)   (–1,–1)

You will note that all 4 of these points fall on our initial line from problem 1. If we take an arbitrary number, say *t,* copies of $\overrightarrow{AB}$, we generate every point on the line containing *A* and *B*. It is this idea that allows us to generate the **vector equation for a line**, which gives a formula in terms of the *x*- and *y*-coordinates of any point on the line:

$(x,y) = (x_1, y_1) + t\langle x_c, y_c \rangle$

where (x_1, y_1) are the coordinates of the initial point and $\langle x_c, y_c \rangle$ are the components of the initial vector.

In our example, the vector equation of the line from $A(1,2)$ to $B(5,8)$ is $(x,y) = (1,2) + t\langle 4,6 \rangle$

You might want to confirm that if you let *t* equal 2, 5, -1, and $-\dfrac{1}{2}$ you will get the coordinates of the points in problem 3, and that each of these points satisfies the equation you found in problem 1 for the line containing *A* and *B*.

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4. Find the vector equation of the line from $B(5,8)$ to $A(1,2)$.

$\langle x,y \rangle = (5,8) + t\langle -4,-6 \rangle$

5. Find the vector equation of the line from $C(-2,5)$ to $D(1,-3)$.

$\langle x,y \rangle = (-2,5) + t\langle 3,-8 \rangle$

You will, no doubt, recall the challenges of solving navigation problems solely by trigonometry and the laws of sines and cosines (a well-drawn picture was truly worth 1000 words). You may also recall that these problems got much easier when we broke them down into *x*- and *y*-components and worked with each separately. There are many contexts in which it is easier to split *x* and *y* apart and consider them somewhat separately (think, for example, of an object flying initially horizontally through the air, subject to gravity and wind resistance in the vertical direction and wind resistance in the horizontal direction). We begin that work, studying *x* and *y* as functions of the **parameter** *t*, now.

The **parametric equations for a line** take the vector equation for that line and split it into its constituent parts. To get the parametric equations for the line from $A(1,2)$ to $B(5,8)$, we start with $(x,y) = (1,2) + t\langle 4,6 \rangle$ and break it into the *x* piece and the *y* piece separately:
$x = 1 + 4t$
$y = 2 + 6t$
Notice that when *t=1*, *x=5* and *y=8*, as desired.

6. Find the parametric equations for the lines given in problems 4 and 5.

4. $x(t) = 5 - 4t$
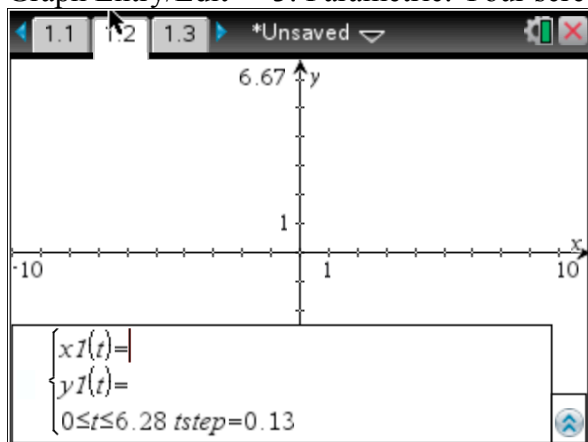 $y(t) = 8 - 6t$

5. $x(t) = -2 + 3t$
   $y(t) = 5 - 8t$

7. Find vector and parametric equations for the line that goes from $E(-3,4)$ to $F(7,6)$.

$\langle x,y \rangle = (-3,4) + t\langle 10,2 \rangle$

$x(t) = -3 + 10t$
$y(t) = 4 + 2t$

8. Find vector and parametric equations for the line that goes from $P(2,5)$ to $Q(-2,-1)$.

$\langle x,y \rangle = (2,5) + t\langle -4,-4 \rangle$
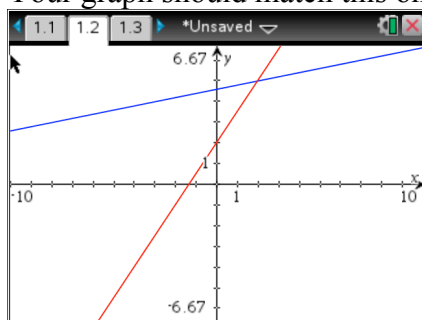
$x(t) = 2 - 4t$
$y(t) = 5 - 4t$

Your calculator can graph parametric equations. Open a graphs page, then choose Menu -> 3: Graph Entry/Edit -> 3: Parametric. Your screen should now look like this:

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9. Use your Nspire to graph the parametric equations for the lines you found in problems 7 and 8. Your graph should match this one:

As you can see, the lines intersect. This guarantees a "path intersection," but we can now concern ourselves with the idea of a "particle collision." The question is now whether a particle traveling according to the first set of equations and one traveling according to the second set will actually collide, or whether they will pass the point of intersections at different times. There are a number of ways to find an answer to this question, but the easiest involves finding the time at which the *x*-coordinates coincide and the time at which the *y*-coordinates coincide and seeing if they are the same:

$$-3+10t = 2-4t \qquad\qquad 4+2t = 5-6t$$

For *x:* $14t = 5$ For *y*: $8t = 1$ Since the simultaneous *x* and

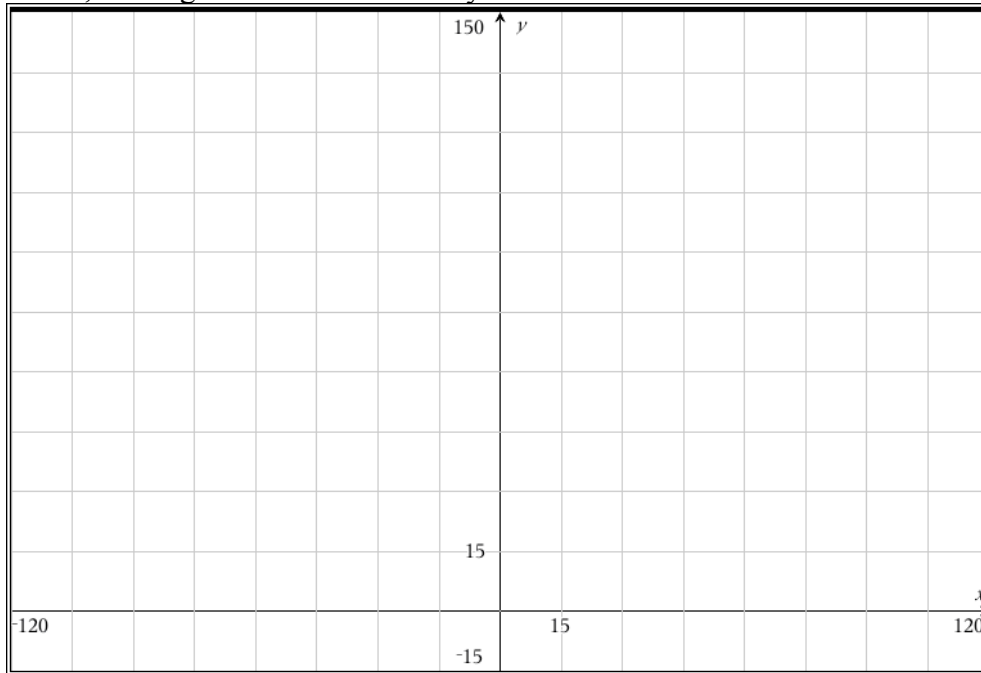$$t = \frac{5}{14} \qquad\qquad\qquad\qquad\qquad t = \frac{1}{8}$$

y occupations happen at different times, these 2 particles do not collide.

APPENDIX H: Unit 10 Activity and Solutions
MA452 Ferris Wheels

Let's consider the London Eye, one of the largest Ferris wheels in the world. The structure is 135 meters tall, and the wheel itself has a diameter of 120m. It turns continuously, completing a single rotation once every 30 minutes. (source: londoneye.com). Can you picture this structure in your head?

Let's place the Eye, in side view, on a coordinate grid to simplify our life. We'll use the ground as the *x*-axis and we'll place the center of the wheel on the *y*-axis. Sketch a drawing on the grid below, making sure to label carefully:

For the moment, let's assume that the loading platform, where you enter the wheel, sits at the same height as the hub at the center of the wheel, in the positive x-direction, and that the wheel turns counterclockwise. The *x*-coordinate of your position, then, starts at its maximum value of 60. Fifteen minutes into the ride, the *x*-coordinate reaches its minimum value of -60. The *x*-coordinate varies sinusoidally with time (not surprising, given that you are moving around a circle).

Write an equation for the *x*-coordinate of a rider as a function of time, with *t=0* when the rider steps onto the wheel from the loading platform:

Describe in words what happens to your *y*-coordinate over time, including the starting value, maximum, and minimum, and the times that max and min are reached.

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The *y*-coordinate, too, varies sinusoidally with time. Use your written description to create a function for the *y*-coordinate as a function of time.

If we pull together the equations for *x*- and *y*-coordinates as parametric equations, we should create the graph of your motion around the Ferris wheel. Write your equations as a pair below, and also write down the values of *t* that you will need to include to generate a full revolution around the wheel. Then use your calculator to graph and see how you did. The trace feature will show you both coordinates and the *t* value simultaneously.

$x1(t) =$
$y1(t) =$
$0 \leq t \leq$ _____

Let's now change the direction of travel. Starting from the same platform to the right of the hub, let's assume the wheel rotates clockwise. Describe how the *x*- and *y*-coordinates now behave (starting values, max and min and when they are reached).

Use this description to write a new pair of parametric equations. Graph and trace again to check your work.

$x1(t) =$
$y1(t) =$
$0 \leq t \leq$ _____

In reality, the loading platform is at the bottom of the wheel. Viewed from one side, motion appears to be clockwise; viewed from the other side, it appears counterclockwise. Write separate pairs of parametric equations to describe clockwise and counterclockwise travel starting from the bottom of the wheel.

Clockwise: Counterclockwise:

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With the loading platform at the bottom and the wheel rotating clockwise, write equations describing your motion if one revolution takes:
15 minutes: An hour:

Look back at all of your pairs of parametric equations. What is in common to all of them? What changes?

Hopefully you spotted some of the following trends:
- Both x and y equations have the same amplitude and period each time. The amplitude is the radius of the circle, so this makes sense, as does the period since the equations describe different facets of the same circular motion.
- The y equations all have a vertical shift up to the center of the circle. If we had put the center somewhere other than along the y-axis, the x equations would also have a shift.
- Each pair of equations has one cosine function and one sine function. The choice of which function is used for x and which for y depends on the starting location along the circle and the direction of travel

$$x1(t) = 60\cos\left(\frac{\pi}{15}t\right)$$

Let's take your first circle, with equations $y1(t) = 60\sin\left(\frac{\pi}{15}t\right) + 75$ and see if we can convert to
$$0 \le t \le 35$$
our more standard, Cartesian equation (one with only x and y as variables). To do this, we will need to utilize the Pythagorean identity relating sine and cosine, namely $\sin^2\theta + \cos^2\theta = 1$. Solve the x equation to get the cosine term by itself and the y equation to get the sine term by itself.

$$x = 60\cos\left(\frac{\pi}{15}t\right)$$
$$\frac{x}{60} = \cos\left(\frac{\pi}{15}t\right)$$

$$y = 60\sin\left(\frac{\pi}{15}t\right) + 75$$
$$y - 75 = 60\sin\left(\frac{\pi}{15}t\right)$$
$$\frac{y-75}{60} = \sin\left(\frac{\pi}{15}t\right)$$

Since $\theta = \dfrac{\pi}{15}t$ for both sine and cosine, squaring each and substituting gives us

$\left(\dfrac{y-75}{60}\right)^2 + \left(\dfrac{x}{60}\right)^2 = 1$. Noticing that the denominators are the same in both fractions, we

typically convert to $(y-75)^2 + x^2 = 60^2$, which fits our general circle equation

$(x-h)^2 + (y-k)^2 = r^2$ nicely.

Using this technique, convert your final pair of parametric equations to Cartesian form.

Surprised? Even though we changed the starting point and the speed of travel, the motion described is still around the same circle.

Problems

1. Describe the motion defined by the following parametric equations, including starting point and speed and direction of travel. Convert each pair of parametric equations to the corresponding Cartesian equation.

(a) $\begin{aligned} x &= 4\cos t + 3 \\ y &= 4\sin t - 2 \end{aligned}$

(b) $\begin{aligned} x &= 5\sin(\pi t) - 1 \\ y &= 5\cos(\pi t) + 2 \end{aligned}$

(c) $\begin{aligned} x &= 3\cos\left(t - \dfrac{\pi}{4}\right) \\ y &= 3\sin\left(t - \dfrac{\pi}{4}\right) \end{aligned}$

2. Write a pair of parametric equations that define motion around the circle of radius 2 centered at (3,4). Indicate the coordinates of your starting point and the direction in which your particle is traveling.

OTHER CONIC SECTIONS

Let's now consider stretching out the horizontal or vertical axis of symmetry, pulling a circle into an ellipse. Let x vary sinusoidally from 1 to 9, starting at its highest value and completing a full cycle in 5 minutes. Write an equation describing this motion.

Let y vary sinusoidally from 3 to 7, starting at 5 and going up, completing a full cycle in 5 minutes. Write an equation describing this motion.

If this pair of equations is describing the same particle parametrically, what shape will be traced? Where is the center? Where are the endpoints of the major axis? The minor axis? Can you write a Cartesian equation for the same curve? (Hint: to convert, use the same Pythagorean identity as you did for circles).

What if your particle is to travel 3 times as fast, and to start at the bottom of the ellipse moving clockwise? Write parametric equations describing this situation.

Write parametric equations for travel around an ellipse centered at (3,2), with vertical major axis of length 10 and minor axis of length 6. Indicate your starting point and the direction of travel.
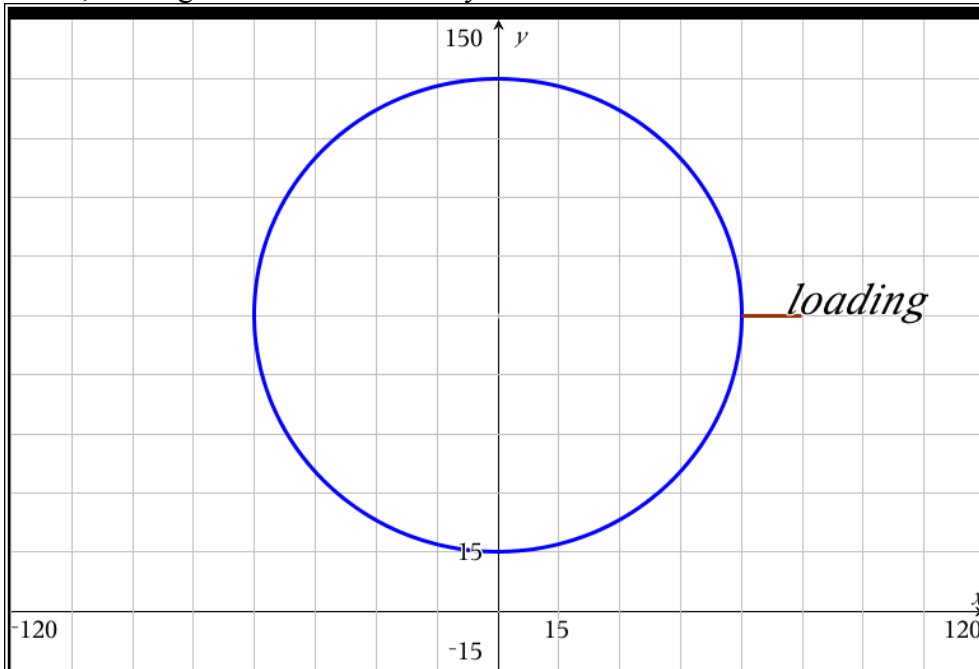
To change from a circle to an ellipse, we let the amplitudes differ between the x curve and the y curve, while keeping the same period. If we also allow the period to vary, we generate a shape called a Lissajous curve. Use your calculator to graph the following parametric curves, each on its own graphs page. Can you figure out what each number is controlling?

1. $x = 3\cos(t)$
   $y = 3\sin(2t)$

2. $x = \cos(3t)$
   $y = \sin(2t)$

3. $x = 3\cos(3t)$
   $y = \sin(2t)$

4. $x = 3\sin(3t)$
   $y = \cos(2t)$

5. $x = 2\sin(2t)$
   $y = 3\cos(3t)$

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MA452 Ferris Wheels

Let's consider the London Eye, one of the largest Ferris wheels in the world. The structure is 135 meters tall, and the wheel itself has a diameter of 120m. It turns continuously, completing a single rotation once every 30 minutes. (source: londoneye.com). Can you picture this structure in your head?

Let's place the Eye, in side view, on a coordinate grid to simplify our life. We'll use the ground as the *x*-axis and we'll place the center of the wheel on the *y*-axis. Sketch a drawing on the grid below, making sure to label carefully:

For the moment, let's assume that the loading platform, where you enter the wheel, sits at the same height as the hub at the center of the wheel, in the positive x-direction, and that the wheel turns counterclockwise. The *x*-coordinate of your position, then, starts at its maximum value of 60. Fifteen minutes into the ride, the *x*-coordinate reaches its minimum value of -60. The *x*-coordinate varies sinusoidally with time (not surprising, given that you are moving around a circle).

1. Write an equation for the *x*-coordinate of a rider as a function of time (in minutes), with *t=0* when the rider steps onto the wheel from the loading platform:

$$x(t) = 60\cos\left(\frac{\pi}{15}t\right)$$

2. Describe in words what happens to your *y*-coordinate over time, including the starting value, maximum, and minimum, and the times that max and min are reached.

You start at a height of 75m, the middle value. 7.5 minutes later, the wheel has rotated one-quarter of a turn and you reach the peak height of 135m. 15 minutes after that, 22.5 minutes into your ride, you are at the low point of 15m.

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3. The y-coordinate, too, varies sinusoidally with time. Use your written description to create a function for the y-coordinate as a function of time.

$$y(t) = 60\sin\left(\frac{\pi}{15}t\right) + 75$$

4. If we pull together the equations for x- and y-coordinates as parametric equations, we should create the graph of your motion around the Ferris wheel. Write your equations as a pair below, and also write down the values of t that you will need to include to generate a full revolution around the wheel. Then use your calculator to graph and see how you did. The trace feature will show you both coordinates and the t value simultaneously.

$$x1(t) = 60\cos\left(\frac{\pi}{15}t\right)$$

$x1(t) =$
$y1(t) =$
$0 \le t \le$ _____

$$y1(t) = 60\sin\left(\frac{\pi}{15}t\right) + 75$$

$$0 \le t \le 30$$

5. Let's now change the direction of travel. Starting from the same platform to the right of the hub, let's assume the wheel rotates clockwise. Describe how the x- and y-coordinates now behave (starting values, max and min and when they are reached).

x still starts at a maximum, with the same amplitude and period. y still starts at the midline, but now it goes down first rather than up, so the times to reach maximum and minimum are flipped.

6. Use this description to write a new pair of parametric equations. Graph and trace again to check your work.

$$x1(t) = 60\cos\left(\frac{\pi}{15}t\right)$$

$x1(t) =$
$y1(t) =$
$0 \le t \le$ _____

$$y1(t) = -60\sin\left(\frac{\pi}{15}t\right) + 75$$

$$0 \le t \le 30$$

7. In reality, the loading platform is at the bottom of the wheel. Viewed from one side, motion appears to be clockwise; viewed from the other side, it appears counterclockwise. Write separate pairs of parametric equations to describe clockwise and counterclockwise travel starting from the bottom of the wheel.

Clockwise:

$$x1(t) = -60\sin\left(\frac{\pi}{15}t\right)$$

$$y1(t) = -60\cos\left(\frac{\pi}{15}t\right) + 75$$

$$0 \le t \le 30$$

Counterclockwise:

$$x1(t) = 60\sin\left(\frac{\pi}{15}t\right)$$

$$y1(t) = -60\cos\left(\frac{\pi}{15}t\right) + 75$$

$$0 \le t \le 30$$

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8. With the loading platform at the bottom and the wheel rotating clockwise, write equations describing your motion if one revolution takes:

15 minutes:

$$x1(t) = -60\sin\left(\frac{2\pi}{15}t\right)$$

$$y1(t) = -60\cos\left(\frac{2\pi}{15}t\right) + 75$$

$$0 \le t \le 30$$

An hour:

$$x1(t) = -60\sin\left(\frac{\pi}{30}t\right)$$

$$y1(t) = -60\cos\left(\frac{\pi}{30}t\right) + 75$$

$$0 \le t \le 30$$

9. Look back at all of your pairs of parametric equations. What is in common to all of them? What changes?

Answers will vary. Important pieces are also spelled out below in the student handout.

Hopefully you spotted some of the following trends:

- Both x and y equations have the same amplitude and period each time. The amplitude is the radius of the circle, so this makes sense, as does the period since the equations describe different facets of the same circular motion.
- The y equations all have a vertical shift up to the center of the circle. If we had put the center somewhere other than along the y-axis, the x equations would also have a shift.
- Each pair of equations has one cosine function and one sine function. The choice of which function is used for x and which for y depends on the starting location along the circle and the direction of travel

$$x1(t) = 60\cos\left(\frac{\pi}{15}t\right)$$

Let's take your first circle, with equations $y1(t) = 60\sin\left(\frac{\pi}{15}t\right) + 75$ and see if we can convert to

$$0 \le t \le 35$$

our more standard, Cartesian equation (one with only x and y as variables). To do this, we will need to utilize the Pythagorean identity relating sine and cosine, namely $\sin^2\theta + \cos^2\theta = 1$. Solve the x equation to get the cosine term by itself and the y equation to get the sine term by itself.

$$x = 60\cos\left(\frac{\pi}{15}t\right)$$

$$\frac{x}{60} = \cos\left(\frac{\pi}{15}t\right)$$

$$y = 60\sin\left(\frac{\pi}{15}t\right) + 75$$

$$y - 75 = 60\sin\left(\frac{\pi}{15}t\right)$$

$$\frac{y - 75}{60} = \sin\left(\frac{\pi}{15}t\right)$$

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Since $\theta = \dfrac{\pi}{15}t$ for both sine and cosine, squaring each and substituting gives us

$\left(\dfrac{y-75}{60}\right)^2 + \left(\dfrac{x}{60}\right)^2 = 1$. Noticing that the denominators are the same in both fractions, we

typically convert to $(y-75)^2 + x^2 = 60^2$, which fits the general circle equation

$(x-h)^2 + (y-k)^2 = r^2$ nicely.

10. Using this technique, convert your final pair of parametric equations in question 8 to Cartesian form.

$$x = -60\sin\left(\dfrac{\pi}{30}t\right)$$
$$\dfrac{x}{-60} = \sin\left(\dfrac{\pi}{30}t\right)$$

$$y = -60\cos\left(\dfrac{\pi}{30}t\right) + 75$$
$$y - 75 = -60\cos\left(\dfrac{\pi}{30}t\right)$$
$$\dfrac{y-75}{-60} = \cos\left(\dfrac{\pi}{30}t\right)$$

$$\left(\dfrac{x}{-60}\right)^2 + \left(\dfrac{y-75}{-60}\right)^2 = 1$$
$$x^2 + (y-75)^2 = (-60)^2$$
$$x^2 + (y-75)^2 = (60)^2$$

Surprised? Even though we changed the starting point and the speed of travel, the motion described is still around the same circle.

Problems

1. Describe the motion defined by the following parametric equations, including starting point and speed and direction of travel. Convert each pair of parametric equations to the corresponding Cartesian equation.

(a) $\begin{aligned}x &= 4\cos t + 3\\ y &= 4\sin t - 2\end{aligned}$

(b) $\begin{aligned}x &= 5\sin(\pi t) - 1\\ y &= 5\cos(\pi t) + 2\end{aligned}$

(c) $\begin{aligned}x &= 3\cos\left(t - \dfrac{\pi}{4}\right)\\ y &= 3\sin\left(t - \dfrac{\pi}{4}\right)\end{aligned}$

(a) This is a circle of radius 4 centered at $(3,-2)$. The motion begins on the right and goes counterclockwise, completing a full revolution every 2π units of time. $(x-3)^2 + (y+2)^2 = 4^2$

(b) This circle has radius 5 and is centered at $(-1,2)$. The motion begins at the top and goes clockwise, completing a full revolution every 2 units of time. $(x+1)^2 + (y-2)^2 = 5^2$

(c) This circle of radius 3 is centered at the origin. The motion starts at the point $\dfrac{\pi}{4}$ radians down from the right and moves counterclockwise, completing a full revolution every 2 units of time. $x^2 + y^2 = 3^2$

2. Write a pair of parametric equations that define motion around the circle of radius 2 centered at (3,4). Indicate the coordinates of your starting point and the direction in which your particle is traveling.

Many possibilities exist. Amplitude must be 2 on both x and y, with shifts of 3 on x and 4 on y.

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OTHER CONIC SECTIONS

1. Let's now consider stretching out the horizontal or vertical axis of symmetry, pulling a circle into an ellipse. Let x vary sinusoidally from 1 to 9, starting at its highest value and completing a full cycle in 5 minutes. Write an equation describing this motion.

$$x(t) = 4\cos\left(\frac{2\pi}{5}t\right) + 5$$

2. Let y vary sinusoidally from 3 to 7, starting at 5 and going up, completing a full cycle in 5 minutes. Write an equation describing this motion.

$$y(t) = 2\sin\left(\frac{2\pi}{5}t\right) + 5$$

3. If this pair of equations is describing the same particle parametrically, what shape will be traced? Where is the center? Where are the endpoints of the major axis? The minor axis? Can you write a Cartesian equation for the same curve? (Hint: to convert, use the same Pythagorean identity as you did for circles).

This is an ellipse centered at (5,5). The endpoints of the major axis are at (1,5) and (9,5). The endpoints of the minor axis are at (5,3) and (5,7).

The Cartesian equation is $\left(\dfrac{x-5}{4}\right)^2 + \left(\dfrac{y-5}{2}\right)^2 = 1$.

4. What if your particle is to travel 3 times as fast, and to start at the bottom of the ellipse moving clockwise? Write parametric equations describing this situation.

$$x(t) = -4\sin\left(\frac{6\pi}{5}t\right) + 5$$

$$y(t) = -2\cos\left(\frac{6\pi}{5}t\right) + 5$$

5. Write parametric equations for travel around an ellipse centered at (3,2), with vertical major axis of length 10 and minor axis of length 6. Indicate your starting point and the direction of travel.

Many possibilities exist: $\begin{array}{l} x(t) = \pm 3\, trigfxn(t) + 3 \\ y(t) = \pm 5\, trigfxn(t) + 2 \end{array}$

To change from a circle to an ellipse, we let the amplitudes differ between the x curve and the y curve, while keeping the same period. If we also allow the period to vary, we generate a shape called a Lissajous curve. Use your calculator to graph the following parametric curves, each on its own graphs page. Can you figure out what each number is controlling?

1. $\begin{array}{l} x = 3\cos(t) \\ y = 3\sin(2t) \end{array}$ 2. $\begin{array}{l} x = \cos(3t) \\ y = \sin(2t) \end{array}$ 3. $\begin{array}{l} x = 3\cos(3t) \\ y = \sin(2t) \end{array}$ 4. $\begin{array}{l} x = 3\sin(3t) \\ y = \cos(2t) \end{array}$

5. $\begin{array}{l} x = 2\sin(2t) \\ y = 3\cos(3t) \end{array}$

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APPENDIX I: Unit 12 Activity
Don't Go Chasing Waterfalls...

Let us assume that Earth's entire water is stored in four distinct reservoirs: as vapor in the atmosphere, as salt water in the oceans, as fresh water in rivers and lakes, and in ground reservoirs such as soil moisture and underground in caves. The illustration below shows a
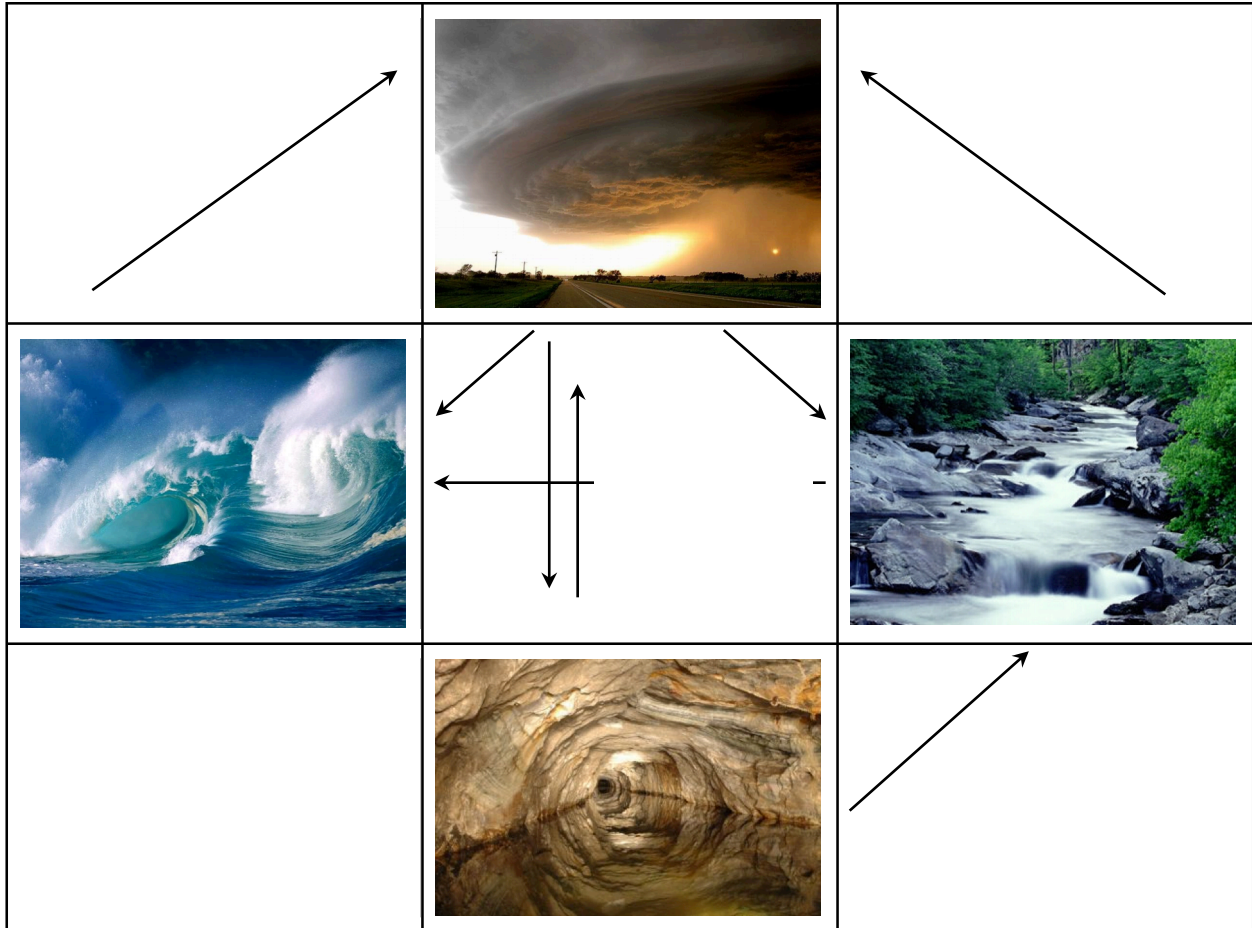
simplified scheme of how water may be transported between these reservoirs.

Exercise 1: Discuss and interpret the above diagram. In particular, think about and explain the *absence* of arrows between certain water reservoirs.

In this project, you will study a mathematical model for the water cycle. Starting with a simplified experiment, we will investigate the math behind the model and convince us of its validity. Then more realistic values can be substituted into the model to predict what is happening on a global scale - a scale which is impossible to recreate in a lab. Below are two sources for realistic data.

- http://www.newworldencyclopedia.org/entry/Water_cycle
- http://ga.water.usgs.gov/edu/waterdistribution.html

Part I: Experimental Investigation

Before studying a suitable mathematical model with actual data, we will first simulate what happens with a simplified experiment. The setup is as follows:

Ground Water Evaporation
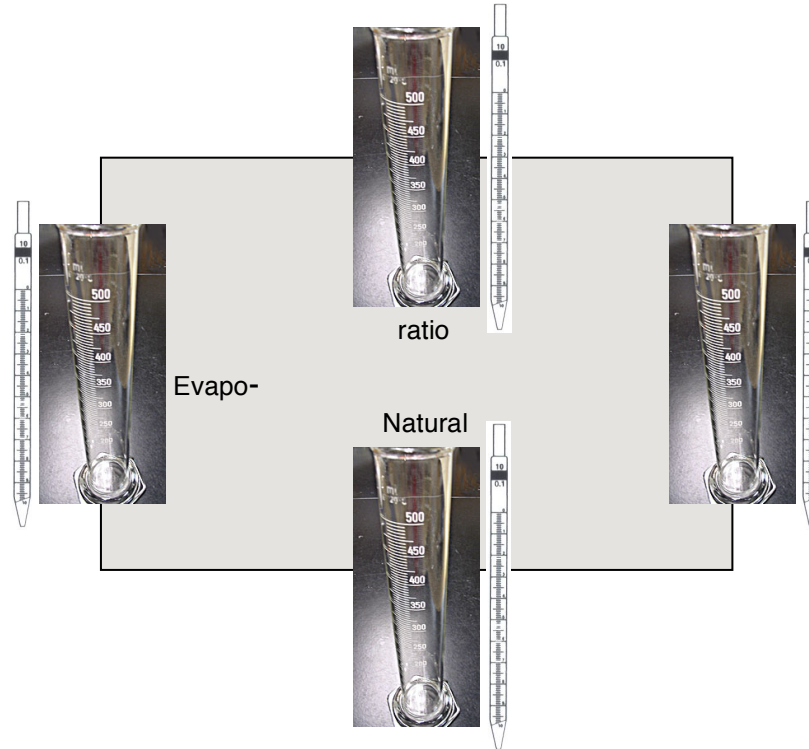
In order to make the experiment feasible, we will work with values for the amounts of water transferred that may not be quite realistic. However, this highlights an important reason for creating mathematical models: *Once the math is set up correctly, experiments can be performed on subatomic or planetary scales without being constrained by the limitations of lab equipment.* Studying such a model on the computer will constitute **Part II** of the lab.

Exercise 2: In plain English, explain how to interpret the entries in the "From Rivers" *column*. What do you notice about the column sums?

In plain English, explain how to interpret the entries in the "To Oceans" *row*. What is the meaning of the zero?

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The steps of the experiment are as follows:

1. Decide who will be responsible for managing each of the four reservoirs: Air, Ocean, River, Cave
2. Distribute 1000 ml of (virtual) water arbitrarily between the reservoirs. In reality, this simply means coming up with four numbers that add up to 1000, and assigning them as initial value to each reservoir. This is the "INITIAL AMOUNT" you will fill into the first field of your worksheet. Please do not use 4x250ml to ensure that each team has different starting values.
3. Perform the experiment until the table is filled in completely:
   i. Based on the transition table and the current water level in your reservoir, determine the amount of water to be extracted and transferred to *each* of the others. Fill those values into the "Give To" fields of the table.
   ii. Use the paper tokens provided to give the appropriate "Give to" amounts to your teammates. (This is faster, cleaner, and more accurate than actually passing water around, but it gives you a physical sense for the process we are modeling.)
   iii. Record the water received from your teammates in the bottom half of the table.
   iv. Add and subtract the changes to obtain the new amount of water. Record it in the bottom field and copy it as your new current value to the top of the next column.

If you know how, you may use the Excel spreadsheet `water_lab_calc.xltx` for the book-keeping. Your teacher can give or send the template to you.

Exercise 3:
1. What trends or patterns do you observe regarding the amount of water in each reservoir as time progresses?

2. Compare your results in the top row to those of other teams. What do you observe?

Part II: Mathematical Model

Rather than calculating the amounts of water step-by-step, we will develop a mathematical model which allows us to predict the amount of water in each reservoir at any time in the future with just one calculation.

The amounts of water in each reservoir (**a**ir, **o**ceans, **r**ivers, and **c**aves) shall be denoted by the variables a, o, r, c.

The amount of water at the next step can now be computed easily using the method you just used for the experiment. For example, consider the change in the water level in the oceans. You get water from each of the other reservoirs, as well as keeping a proportion of your own.

Exercise 4: If the current amount of water in the oceans is o now, and the current amount of water in the atmosphere is a, in rivers and lakes r, and ground water c, write expressions for how much water

• stays in the oceans:

and the amounts you receive from:

• the atmosphere:
• the rivers and lakes:
• from ground water:

Determine the new amount of water in the oceans by <u>adding the water remaining in the oceans and the</u> <u>contributions</u> <u>from the other</u> <u>sources</u>:

Precipitation

Without the intermediate steps, write the other three equations as well:

Rivers & Streams

Using the subscript n to indicate "current value" at time n, and the subscript $n+1$ (the next number after n) to symbolize the value after the changes, we can write the "arrow equations" in

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a more standard way. For example, the amount of water in the oceans at time $n+1$ is calculated from the other amounts as follows: $o_{n+1} = 0.25a_n + 0.85o_n + 0.2r_n + 0c_n$.
You can now set up all four equations in this format, and, by stacking them, write the mathematical model of the water system as one matrix equation:

$$\begin{bmatrix} a_{n+1} \\ o_{n+1} \\ r_{n+1} \\ c_{n+1} \end{bmatrix} = \begin{bmatrix} 0.60 & 0.15 & 0.10 & 0.05 \\ 0.25 & 0.85 & 0.20 & 0.00 \\ 0.10 & 0.00 & 0.70 & 0.05 \\ 0.05 & 0.00 & 0.00 & 0.90 \end{bmatrix} \begin{bmatrix} a_n \\ o_n \\ r_n \\ c_n \end{bmatrix}$$

where the vector

$$\begin{bmatrix} a_0 \\ o_0 \\ r_0 \\ c_0 \end{bmatrix}$$

is the INITIAL AMOUNT vector of the system - the amount of water you chose for each reservoir at the beginning.

Exercise 5: Write out the matrix multiplication and compare the resulting equations to the ones you found in **Exercise 4**.

Suppose w_n is the 4x1 "water vector" at time n, and T is the 4x4 matrix telling us how how to get from w_n to w_{n+1} (called *transition matrix*). Then we can write, even more succinctly, $w_{n+1} = T w_n$, with w_0 containing the four initial conditions.

Write the amount of water, in milliliters, initially in each reservoir a, o, r, and c in a column vector, and call it w_0.

$$\mathbf{w}_0 = \begin{bmatrix} a_0 \\ o_0 \\ r_0 \\ c_0 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

To obtain w_1, multiply T by w_0. (Note how in this step $n=0$ and so $n+1=1$.)

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Exercise 6: Repeat the experiment using your calculator and matrix multiplication. Record your results in the table below and compare with your experimental values.

	Amount in (ml)			
	w_1	w_2	w_3	w_4
Air				
Oceans				
Rivers				
Caves				

Exercise 7: Show the following equations algebraically:

$w_1 = T\, w_0$ (This is clear because that's how the model is defined.)

$w_2 = T^2\, w_0$

$w_3 = T^3\, w_0$
...

and, in general,

$w_n = T^n\, w_0$

Exercise 8: Use the last equation to recalculate your final values for the water vector w_8 in one step.

Water has been circulating on Earth for Millions of years. This is not something we can simulate in the lab; not even something we want to calculate step-by-step. However, with our mathematical model, we can easily study the long-term behavior of the system.

Exercise 9: Determine w_{200} and w_{201} for three different choices of w_0. Use the initial values chosen by other groups, for example, or make them up. (If you do make them up, what constraint do you have to observe for the entries of w_0?)

$$\mathbf{w}_0 = \begin{bmatrix} & \\ & \end{bmatrix} \qquad \mathbf{w}_{200} = \begin{bmatrix} & \\ & \end{bmatrix} \qquad \mathbf{w}_{201} = \begin{bmatrix} & \\ & \end{bmatrix}$$

$$\mathbf{w}_0 = \begin{bmatrix} & \\ & \end{bmatrix} \qquad \mathbf{w}_{200} = \begin{bmatrix} & \\ & \end{bmatrix} \qquad \mathbf{w}_{201} = \begin{bmatrix} & \\ & \end{bmatrix}$$

$$\mathbf{w}_0 = \begin{bmatrix} & \\ & \end{bmatrix} \qquad \mathbf{w}_{200} = \begin{bmatrix} & \\ & \end{bmatrix} \qquad \mathbf{w}_{201} = \begin{bmatrix} & \\ & \end{bmatrix}$$

1. What do you notice? How does this reflect the behavior of your experimental system?

2. Comparing the values of w_{200} and w_{201} in particular, what do you conclude about the behavior of the system in the long run?

3. (*) Can you think of an equation which can predict the water vector as $n \to \infty$?

Exercise 10:

We want to solve the equation

$$\begin{bmatrix} a \\ o \\ r \\ c \end{bmatrix} = \begin{bmatrix} 0.60 & 0.15 & 0.10 & 0.05 \\ 0.25 & 0.85 & 0.20 & 0.00 \\ 0.10 & 0.00 & 0.70 & 0.05 \\ 0.05 & 0.00 & 0.00 & 0.90 \end{bmatrix} \begin{bmatrix} a \\ o \\ r \\ c \end{bmatrix}$$

for the unknowns *a*, *o*, *r*, and *c*, which satisfy $a + o + r + c = 1000$. This last equation can be tagged on to the first four.

$$\begin{bmatrix} a \\ o \\ r \\ c \\ 1000 \end{bmatrix} = \begin{bmatrix} 0.60 & 0.15 & 0.10 & 0.05 \\ 0.25 & 0.85 & 0.20 & 0.00 \\ 0.10 & 0.00 & 0.70 & 0.05 \\ 0.05 & 0.00 & 0.00 & 0.90 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ o \\ r \\ c \end{bmatrix}$$

Interpret the meaning of the solution to this system in the context of the problem.

Exercise 11: The solution to this system of equations is called the "*steady state*" of the system. Explain why this term is appropriate.

Use a separate piece of paper to work on any of the following, as assigned.

Problem 1: Imagine we were somehow able to tag one of the water molecules and follow its fate. What does the above mathematical model tell us about the location of the molecule at each time step *n*? Write a few sentences outlining your analysis and interpretation.

Problem 2: Use the resources listed at the beginning to set up the water model with realistic values. Study the long-term behavior of the model. Study the long-term effects of "mining water" (extracting water from underground resources), and the effects of melting polar caps (more fresh water flowing into the oceans).

Problem 3: Discover or invent a similar system and investigate its long-term behavior. Does it have a steady state like the water problem?

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Appendix J: Student Feedback

These are student answers, unedited. As these activities were completed early in the year, I expect that the written reflections will improve as the year goes on, and that the reflection itself will be a valuable exercise for the students. Their answers to what they do and don't understand certainly inform my wrap-up lesson about each activity and what we come back to over time. Two class sections answered the questions about the dynamical systems activity, while only one section (mine) answered the questions about the fractal coastline activity.

A. DYNAMICAL SYSTEMS ACTIVITY

1. What big ideas did you take away from the lab?

How to find fixed point graphically and algebraically. How to plot and analyze cobwebs.

How to connect data to a graph, how to analyze data in a different way, how to see patterns and explain them algebraically.

I learned how patterns in equation and graph can change in a dramatic way if a little bit of equation or information is changed.

We can use iteration-output and input-output graphs together to try to find a pattern.

We reviewed the idea of the difference between recursive and explicit modeling which was confusing for me last year. I also took away a much better understanding of cobweb diagrams. Also, I am much more comfortable with the "fill" function on the calculator.

This lab was a very good transition from the drug lab. I found it most interesting at the end when we discovered which R values broke the pattern. This lab helped me to learn how to better analyze functions and their patterns.

Visualizing patterns in functions based on their cobwebs.

Logistic functions, cobweb, and the different ways that your prediction can not work. I have never seen cobwebs before and it was a little confusing at the start but I understood it later.

I learned about the effect of the r-value on an equation through this lab.

I realized that there are a lot more to cobweb diagrams than I thought from last year. That is a little concerning because that could be true for a lot of other concepts I've learned in math.

I learned the value and usefulness of the cobweb diagram as it can represent the long-term outcome of a logistic function due to its relation and intersection with the y=x function.

How to find where a graph stabilizes, how iterations work, how to solve a system of equations, how to cobweb.

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I understand how iterating functions works, and I used input/output (aka cobweb) functions. Furthermore, I learned about time-series graphs.

I really learned that a graph can fluctuate around a number and how to make recursive functions.

Overall, I furthered my understanding of cobweb diagrams and recognized their importance and use.

Visually following an input/output relationship. How the growth rate affects the cobweb function. How to find multiple iterations quickly on my calculator. Better understand how cobwebs are formed.

I saw that there can be very irregular growth/decrease rates.

How to utilize the calculator to plot data. Cobweb diagrams are useful for seeing trends. Dynamical systems are a relationship of two or more quantities that have a fixed rule to describe it.

The first lesson that struck me was the usefulness of the explicit model of the function in comparison to the recursive model. Another big idea was that the limit of the graph is reached more quickly if the r value is higher, and that the system of the function can oscillate between 2, 3, and even 4 values as the r increases. The fact that the system exhibits an indiscernable pattern at a certain point of r was also an interesting idea. The use of cobweb diagrams to find/estimate equilibrium points. In chaos, we can determine the next step with the formula, but we cannot predict the second step and onwards. The input produces the whole range of outputs.

How to graph and analyze a cobweb diagram. The differences between linear and non-linear functions when iterating on a cobweb diagram. Better use of the calculator.

The main concept that I took away from this lab was the use of cobweb diagrams and iteration plots in determining stabilizing points and points of equilibrium. This was relatively easy to visualize using the applet, which gave me an improved understanding of the concept. I also learned more about horizontal asymptotes, especially how they represent stabilizing points. I found the discussion of r value most interesting; in particular, I enjoyed learning about the relationship between r value and asymptotes and stabilizing points.

Cobweb graphing. Carrying capacities. Changing growth factor. Graphing on the calculator. Input/output. Iterations. Stabilization.

How to do cobwebs and how they function. How the rate of change for the cobweb changes the shape of the final graph. Once the rate got above a certain point, though, it confused me.

How to graph cobwebs and where their shape comes from.

I learned that values can fluctuate even once they've leveled out. While looking at cobweb plots helped, I'm still unsure as to how I can graph them on my own without the applet.

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I took the concept of logistic growth from the lab. I thought the lab was very helpful for a biology test I took yesterday where we were using the same logistic curve to model population growth.

2. Did anything surprise you as you went through the lab?

Some of the work was a little more challenging than I thought.

I was surprised when I discovered the cobweb diagram that got smaller as it reached the curve.

The cobweb did not look the way I thought it would. The time it took to plot the input/output graph. How the r value has a key role in how the function behaves.

I expected the graphs to look fairly normal, but they did not.

Nothing surprised me as I went through it.

The thing that surprised me was that the y value of the function could stabilize at a lower point than where it started. I had always expected the y values to stabilize after a steady increase. The concept of an oscillating sequence was also a surprise that I could only vaguely grasp.

Some dynamical systems have no pattern.

I found the methods of algebraic solving that we employed on page 7 a bit surprising. The process was quite counterintuitive at first, but on reflection it makes sense because parabolas have two solutions for a given output, and the plots had 2 stabilizing points. I also found the correlation between the r value and the number of stabilizing points interesting. I found that as r value goes from 0 to 3, the plot approaches one single stabilizing value, from 3-3.5 it oscillates between 2 values, and beyond 3.5 I could find no pattern.

Pattern of graphs. Growth rate changed and was not constant.

Not really, except for how much of the packet I could do on my own.

I don't think I understood enough to be surprised. I had nothing to compare my results with; I wasn't here last year and have not done the drug lab.

It surprised me to see that the values fell between a parabola and y=x rather than between 2 linear functions.

I was surprised by the iteration-output graphs, which showed many relationships which I did not expect.

I was surprised by how the rate and starting point affected the graph, especially the change in the fixed point.

The very last part with r=3.5 definitely surprised me.

The way the equation and pattern change even with the problem which seemed simple, surprised me by a lot.

Yes, I have seen some really amazing graphs. Also, I was impressed because having another starting point or equation can make a big change to the graph.

I was surprised by the idea of the cobweb diagram not approaching a fixed point, like in problem 11 in the second packet. I was also surprised by how much I remembered from last year.

The biggest surprise to me was the pattern being broken at the end of the lab. I was not expecting the cobwebs to become drastically "abnormal" as the r values changed. While I found this surprise interesting, I would have liked more details on WHY this occurred.

A lot of things surprised me in this lab…I had no idea what restrictions or different values can do to a graph. There were some changes that made drastic differences in the appearance of the graph.

One thing that surprised me was that there are spreadsheets and the graphing of spreadsheets on the TI-Nspire. I have owned one for a year and only now have I found out about it.

Not really. A lot of this lab was unclear to me going into it, so there was nothing that I really thought I knew and then was surprised by a different result than I was expecting.

I was surprised that we used the applet because last year we only graphed by hand, but this was much easier.

I was surprised and interested to discover that the lines between the function and the y=x function do not just bounce from one line to the other, but rather the lines are closing in on the intersection point.

How drastically your perception of a graph can change with zoom.

I found that cobwebs were actually quite easy after getting used to them.

3. What ideas are you still confused about?

I am still a little confused by the r-values that caused the graph to oscillate on the intersection point. I'm not sure why the value won't stabilize.

Could use more information on what a cobweb is and why it works, what it represents.

I am still wondering why did the graph change like this with an equation that doesn't seem hard, and why the equation has so many changes when it contains different values.

How to find the function of iteration. Sometimes it is a little bit complex.

I am still confused about how to convert a function in recursive form to explicit form.

I am slightly confused on the cobweb's value. While I understood the graph and correlation to data, I have trouble comprehending the importance it has. A graph that stabilizes, I think, is just as valuable, if not more valuable than a cobweb.

I am still confused by discrete dynamical systems and some analysis based on the graph.

I think that I understood it all by the end of the lab.

Cobweb graphs. I still don't understand the relevance of them.

I'm still confused on the definition of a discrete dynamical system. I think we should go over it more in class. It's a big idea, but I don't think I really understand it.

I understand but need practice for my ability to algebraically determine an explicit function based on a represented variation in data.

Predicting long-term behavior based on r-value.

I am not particularly confused about anything, but I wish that we had spend a bit more time with time-series graphs.

How to solve for equations algebraically for a stabilizing value. Also how to predict when a graph will stabilize.

I am confused about the cobweb diagram with the curve and solving algebraically.

Page 7, wasn't sure how to solve algebraically. Page 8, the question about how many values there are in respect to the value of r. Does stabilizing value mean at what point all the values cease to change?

I'm still confused about how to algebraically solve for the equations. I can make little sense of it.

I'm still a little confused on what the use case for dynamical systems are.

I'm confused about the systems that oscillate and those that portray indiscernable values. I would like to learn more about the relationship of 'r' and the patterns of the graphs.

Dynamical systems. Oscillations. Algebraic work to predict stabilizing values.

I'm still a bit confused about why these patterns occurred with different r values. I am curious to know exactly why I discovered the correlation between r-value magnitude and the number of stabilizing values. In addition, I am confused about the use of Data and Statistics pages, because they did not plot values properly for me. Finally, I am unsure of whether I was supposed to find exact stabilizing values on page 8 or simply estimate graphically.

Solving algebraically for the stabilizing value. What a discrete dynamical system is. Oscillations. Continuous graphs.

Definition of a dynamical system, and that I am so confused about, like I have no idea how to begin.

What is the point of a cobweb function? I also don't understand why entering a growth rate larger than 1 in the iterated linear model affected the graph the way it did.

It's still unclear how I can solve algebraically for a stabilizing value. Also, predicting stabilizing values is still unclear.

I am very confused how some of the relationships appear to have no patterns. I am mildly confused about why these functions don't have an equilibrium point after 'r' is changed.

4. What parts of the reading in the handout were unclear?

What the definition of a dynamical system was.

It asked us to write observations, but never made it clear what conclusions we should draw from these observations.

The wording on creating the graph with iterations and output.

I wanted more about the definitions. I had to think a lot about algebraically solving stuff. Also, it took a while to understand what the reading was asking me in all of section C.

The continuous case. I'd like to know more about deriving an explicit function. I want someone to give me a definition of a discrete dynamical system.

I wanted more information on the last section of reading. However, the reading was generally very clear and straightforward.

I want to know more about the difference between linear and non-linear functions when iterated. I want the real definition for a discrete dynamical system, not something I found online.

The reading was generally very clear and straightforward, and the instructions for the calculator were very clear overall.

I thought that Part C pages 7 and 8 were the hardest.

I did not understand the instructions for how to solve algebraically.

Page 8, the paragraph about finding discernable patterns and oscillation of values could have had more examples.

I wish I had more information on the calculator parts, as well as solving algebraically.

Part C, the last reading on oscillating and finding patterns.

I found part C slightly confusing due to the length of some of the questions. I was having a bit of trouble keeping up with the steps required.

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Packet did a nice job of adding from the previous questions in a very structured way so I was never very confused (only at first with the definition of recursive and explicit equations).

It was unclear at first about using the graph, chart, and calculator.

It is very clear.

Nothing was really unclear in the handout. If something was a little confusing, we went over it in class.

While most of the instructions were quite clear, #8 was quite confusing. With a teacher in class it is fine, but I was confused doing it for homework, since I had not been exposed to that type of cobweb.

The vocabulary and loads of information was a bit overwhelming, so it was difficult for me to filter through that in order to get a solid idea of what I had to do.

I think it was unclear when we were asked to solve the functions algebraically; I did not know what to do until it was explained in class.

Most of the readings about the calculator were unclear to me at the start. Also, anything about cobweb diagrams.

The only thing I think was unclear was the part with the definition of a discrete dynamical system.

I thought the readings for the review of recursive definitions in the start of the packet was a little confusing to read. The calculator instructions and the guided questions throughout the rest of the packet were excellent.

Not really anything in particular. Maybe some clarification in how much work we are to do.

There were no particularly unclear areas of the handouts. However, I wish that we had worked more on finding the equations, not graphing the cobweb and time-series graphs.

B. FRACTAL COASTLINE ACTIVITY

1. What big ideas did you take away from the lab?

I learned about fractals and their value in the modern world and also their value in calculating not perfect geometrical shapes. When I looked up fractals in nature it appeared that there were large numbers of natural fractals which are unable to be measured by euclidean geometry.

In the lab, I learned about fractals in the real world and their applications, as well as how they worked and their usefulness in mathematics.

I learned that the greater possible maximum for a function, the larger the power or rate of change will be in the negative direction.

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I learned what fractals were. I also learned how to use logarithms and apply them to real life situations as opposed to just problems in math.

I learned how fractals work, and the relationship between exponential functions and their graphs better. So how to tell which functions are which, and how to translate data. That functions can be transformed and that by looking at residuals it is made more evident which functions match a data set.

Roughness changes the max potential for randomness. Fractals have many important real life applications.

Transforming between power, linear, and exponential equations. Using data and statistics pages on calculator. Using residuals. Regression.

I learned about the broad use of fractals and how they can be applied to certain things. I also learned that the powers in the regression are the same as the slope in the linear equation.

I learned the relationship between a power regression and a linear function of the natural logs of the x and y values. I also learned about the composition and use of fractals.

The main idea that I took away from this lab is how calculators produce regression models to fit data. Previously, I had almost no conception of how it computed the best-fit line. The lab taught me that the regression process involves conversion to linear equations using the properties of logarithms. Additionally, I learned about two of the main indicators of a good fit. Firstly, you must consider the size of the values on the residual plot's y-axis, looking to minimize their magnitude. Secondly, you must look for a "noisy" result with no discernable pattern in the plotted points. Another take away was the role of fractals in everyday life. I learned that fractals are natural phenomena that display a repeating pattern, and that they have applications in studies on carbon content, cancer research, cell phones, and neuroscience, among other things.

2. Did anything surprise you as you went through the lab?

I was surprised to see that we were doing the linear regression then turning it back into a power regression, this seemed redundant to me as while learning about what our calculator does may be important to the curriculum, it also was not needed as it was the equivalent of simply taking a natural log of an equation.

Much of the lab was a surprise. I had no idea what we were getting into when we started, but I must say that I did enjoy it and watching things come together at certain points in the activity.

I was surprised to learn how you may determine which type of function is best fit for your specific data.

Besides how difficult the lab was, nothing that I found seemed to be out of the ordinary. We did however have some points that didn't make sense. One thing that was surprising I must say, was that the points lined up almost exactly with the linear line at one point. I could appreciate how rare that happens in math even without knowing what its significance was per se.

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It was surprising how easy it was to transform one function into another. As well as how the roughness affected the number of possible data points.

Fractals being used in cellphone antennae was surprising, but other than that not much.

Calculator can transform equations and give important info in data and statistics page.

I was surprised to learn how you can determine which type of function will fit your data.

I was surprised about how the linear equation's slope was the power in the power regression.

I was quite surprised at the degree to which fractals are present in everyday life. Before completing my background research, fractals seemed like an entirely abstract concept, but now I see that they are essential components of science, technology, and even art. The calculator's method of calculating regressions also surprised me, and I still find it slightly confusing.

3. What ideas are you still confused about?

I am confused about the linear regression problem, though this is explained above. I am also confused about the value of this lab, while we may see that fractals are a part of nature we also did not go in depth about how they are used in nature and instead learned about how they apply to an applet. This did not render as easy comprehension as going in depth into their uses would have done. I do understand that the lab was valuable for our apprehension of logarithms, though it seemed like we hit an epiphany about fractals and decided to not carry on with learning about them.

I am still a bit confused about the natural log of y=mx+b, and I still need to drill into my mind the differences between the calculated power equation and the power regression, as well as their uses.

I am still confused as to how this related to fractals. I understand the aspect of compounding and maximums, but how they relate in math is still unclear.

Power Regression and exponential power equation and what their significance was.

I'm still confused on the different types of best lines of fit. I understand median-median but am not totally sure why the other was important.

I would like greater detail on how regressions are performed by the calculator. I understand superficially, but I want a more in-depth understanding. I also want to know how calculators perform regressions on sinusoidal and harmonic functions.

I am still a little confused on transforming and seeing the linear pattern.

Making proper transformations between exponential, power, and linear graphs; anticipating what the graphs will be like.

I am still confused how to transform back to determine the calculated power equation.

I'm still confused about the concept of fractals and fractal dimensions.

4. What parts of the reading in the handout were unclear?

The parts of the reading in the lab that were unclear were the portions which explained why learning about how our calculator works is important to our understanding of logarithms. The regression data and portions concerning regression and statistical analysis was valuable though I don't see the value of simply learning about this through a programming situation.

Some of the calculator directions in the lab were hard to comprehend. Also, I, for some reason, had difficulties trying to input the natural logs of "boxcount" into the chart. Also, as I was very busy last night with a massive project and planning for a potential co-curricular project, I do not think that I did the lab to my full potential.

Some of the directions were a bit unclear.

The short snippet of reading on page 5 about 2-dimensional, 1-dimensional and how fractals fall somewhere between was a bit difficult to understand without a bit more research.

Most of the handout was pretty clear, but the section in part C where we had to use R was slightly vague. Going back over my work I had to research to find out what r and r^2 were and then continue from their.

Transformation reading was a little unclear. Difference between power regression and the calculated power equation was a little confusing.

#11, transforming back from linear to power equation.

Parts on determining the power equation were unclear.

The instructions for question 10 were a bit confusing. My partner and I were originally mistaken in finding a power equation for the graph ln(boxcount) vs. ln(gridsize) instead of just boxcount vs. gridsize. We realized our mistake in class after writing them on the board and had to re-calculate all of the power equations. Also, the explanation of fractal dimensions was confusing.

The handout was very clear overall. The only part that confused me was problem 9. This is because I could not tell whether we were expected to calculate numerical values or just state the answer in a generalized form.